

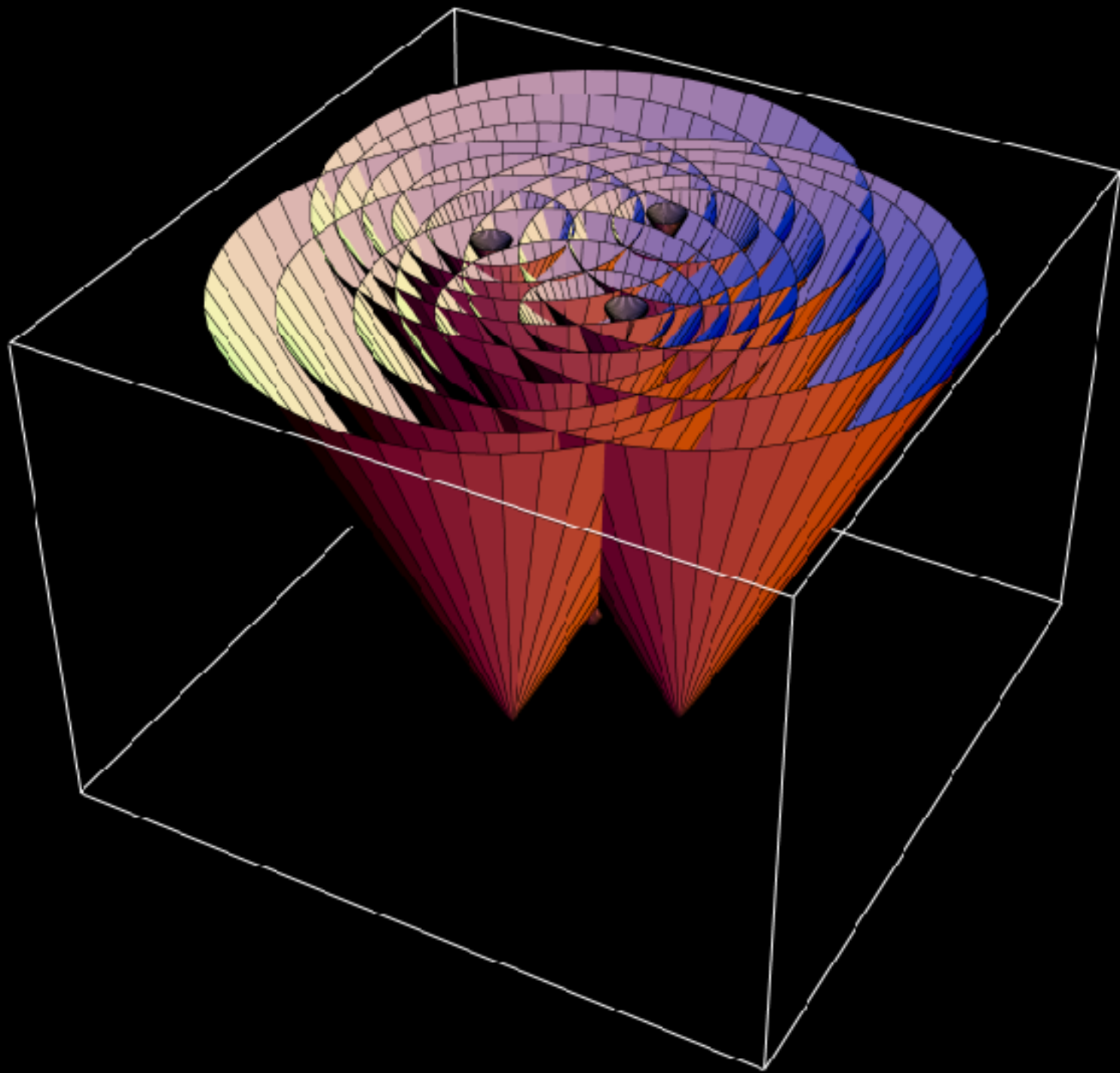
Gravimetry, Relativity, and the Global Navigation Satellite Systems

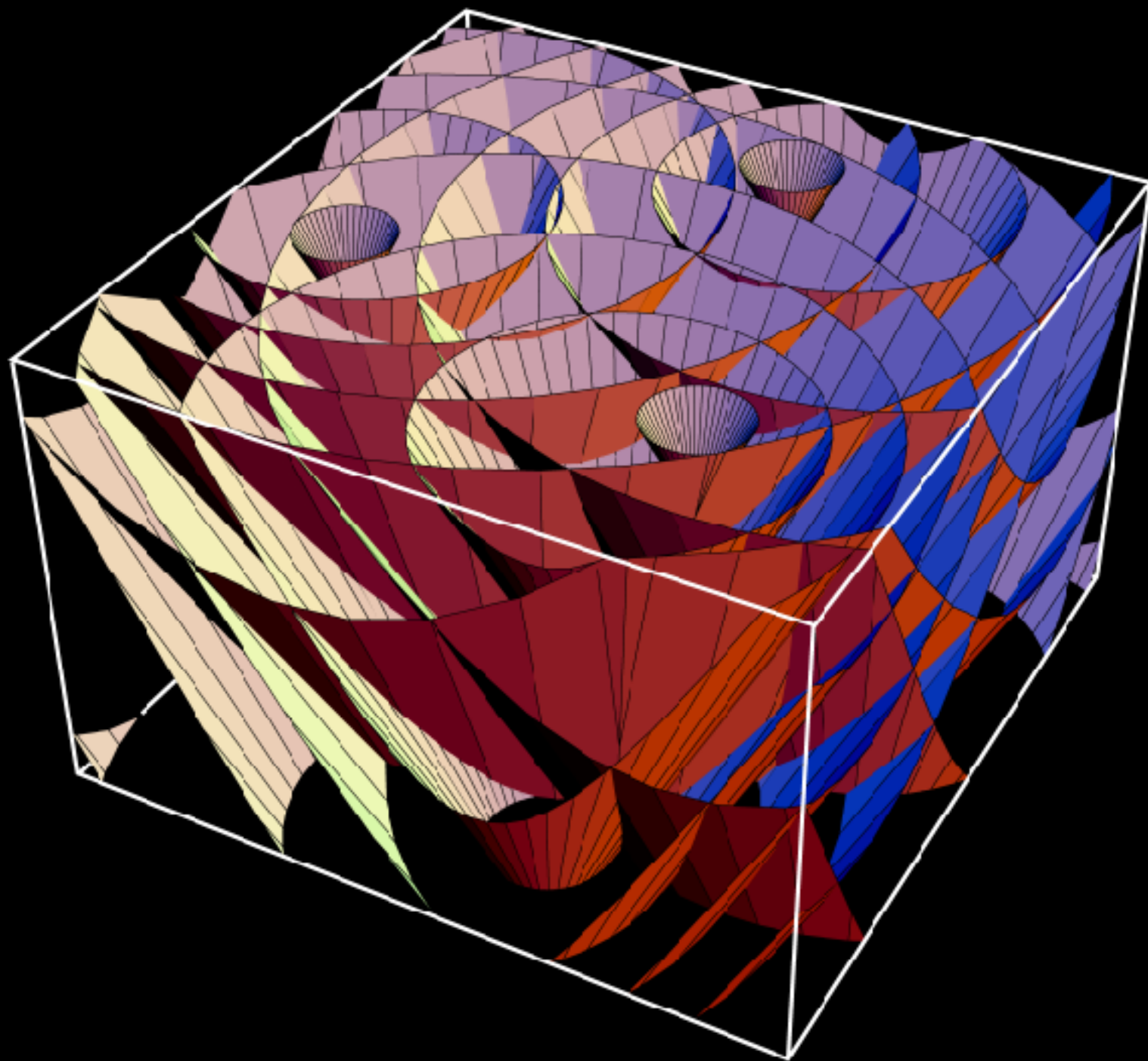
Albert Tarantola

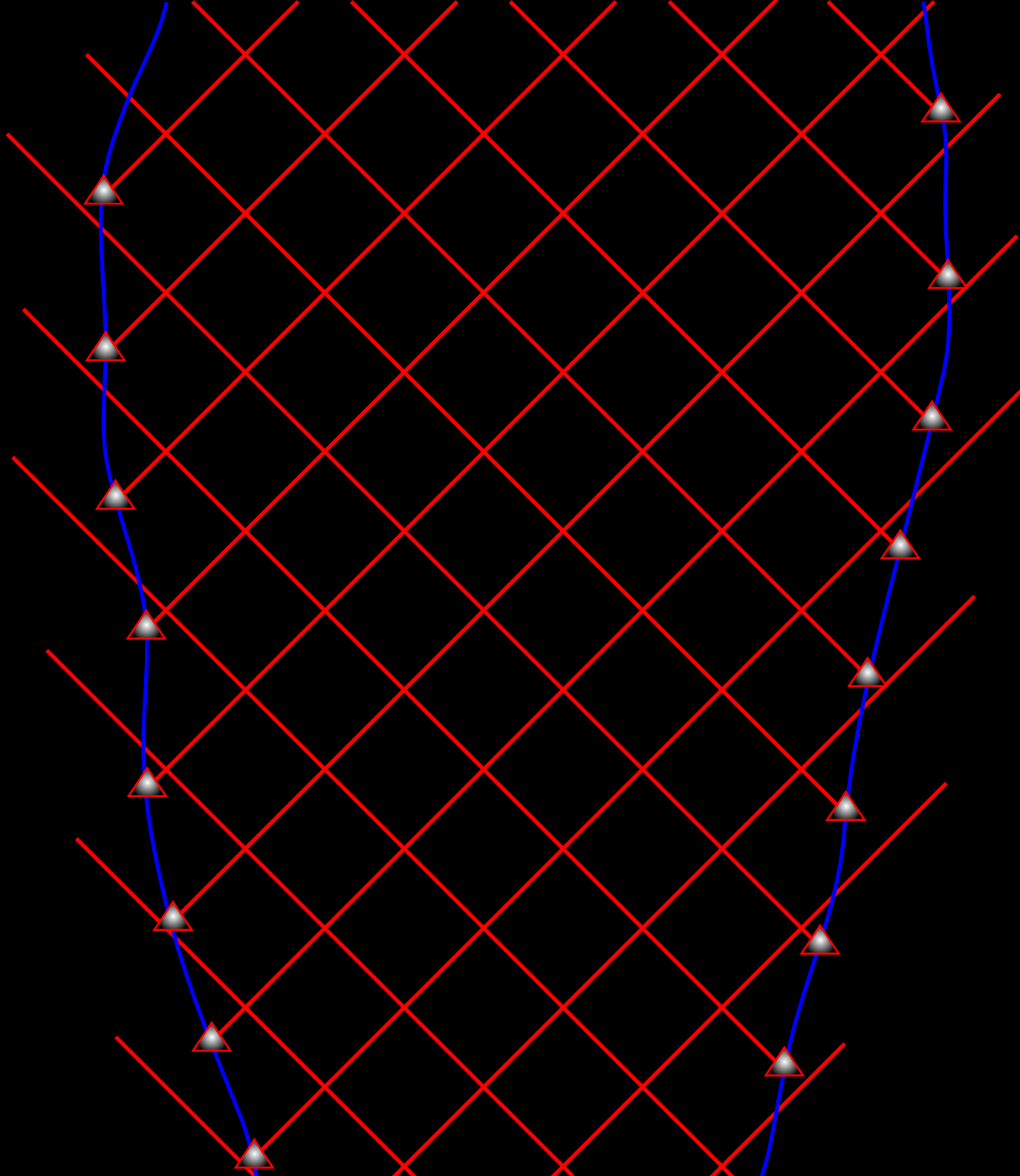
Institut de Physique du Globe de Paris

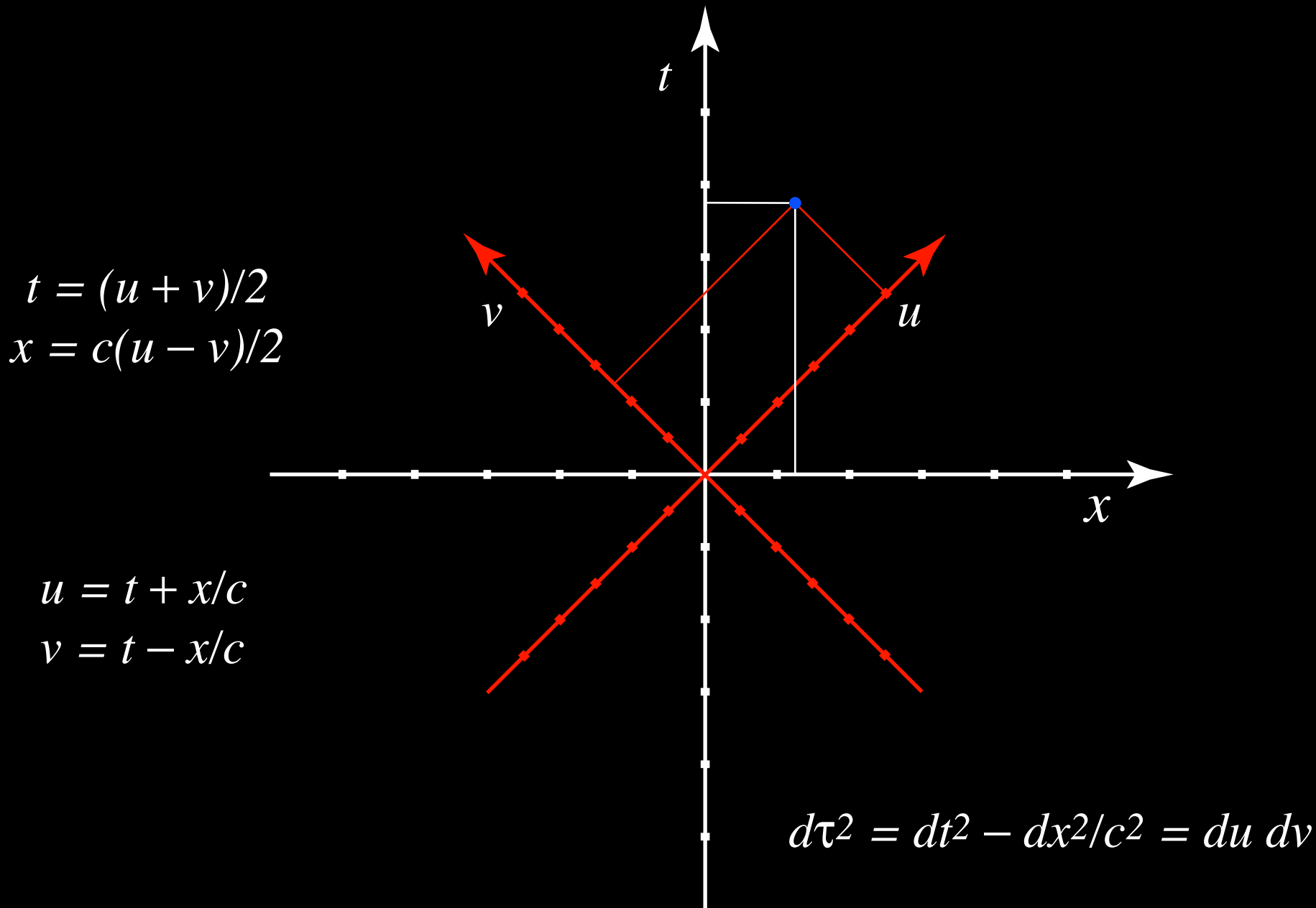
Bartolomé Coll

Observatoire de Paris









Minkowski coordinates

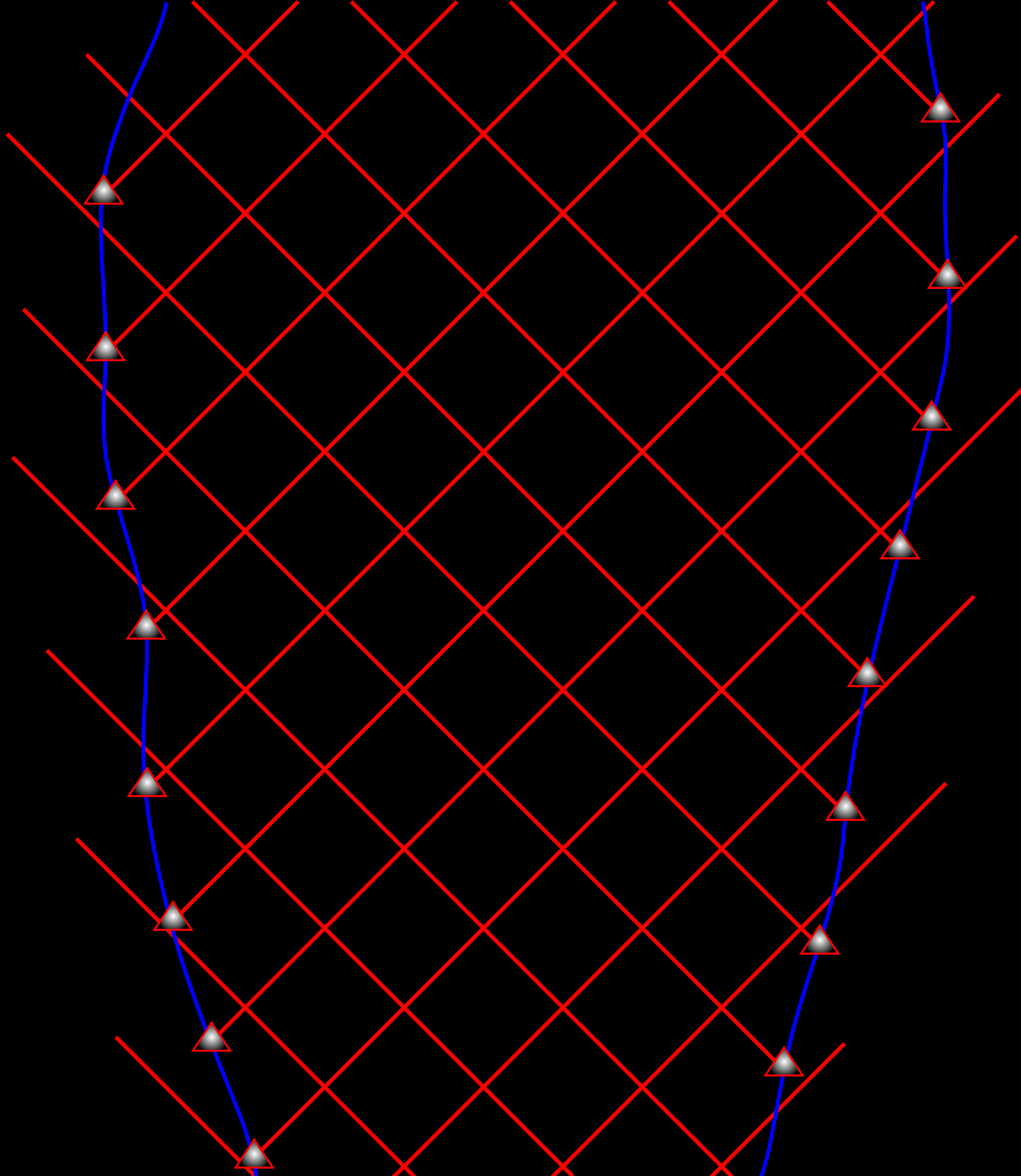
$$d\tau^2 = dt^2 - dx^2/c^2$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1/c^2 \end{pmatrix}$$

Light (or “null”) coordinates

$$d\tau^2 = du \, dv$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$



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Physical Immediate Coordinates (2 space-time dimensions)

$$d\tau^2 = \mu(\tau^1, \tau^2) d\tau^1 d\tau^2$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} 0 & \mu(\tau^1, \tau^2)/2 \\ \mu(\tau^1, \tau^2)/2 & 0 \end{pmatrix}$$

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Physical Immediate Coordinates (4 space-time dimensions)

$$d\tau^2 = g_{\alpha\beta} d\tau^\alpha d\tau^\beta \quad ; \quad (\alpha, \beta, \dots = 1, 2, 3, 4)$$

$$\{g^{\alpha\beta}\} = \begin{pmatrix} 0 & g^{12} & g^{13} & g^{14} \\ g^{12} & 0 & g^{23} & g^{24} \\ g^{13} & g^{23} & 0 & g^{34} \\ g^{14} & g^{24} & g^{34} & 0 \end{pmatrix}$$

Setting of the Problem

Four principal clocks broadcast their proper time. Any observer in space-time receives, at any point along its space-time trajectory, four times $\{\tau^1, \tau^2, \tau^3, \tau^4\}$, that, by definition, are the space-time coordinates.

If the observer has his own clock, with proper time denoted σ , then he knows his **trajectory** $\tau^\alpha = \tau^\alpha(\sigma)$ and his **four-velocity** $u^\alpha = d\tau^\alpha/d\sigma$. The observer may have embarked **accelerometers, gradiometers, and gyroscopes** in his 'satellite'. How can we **estimate the space-time metric** using these data?

A Priori Constraints

The space-time metric shall be determined thanks to different kinds of data, and thanks to three different constraints. The constraints are:

- the diagonal components $\{g^{11}, g^{22}, g^{33}, g^{44}\}$ of the contravariant metric must be zero in the natural basis associated to the coordinates $\{\tau^\alpha\}$;
- the metric has to approximately satisfy the Einstein equations;
- among all possible space-time metrics consistent with our data and other constraints, we wish that the metric is 'smooth'.

Different Kinds of Data

- The signal emitted by one clock at proper time ρ reaches some other clock at proper time σ ; the space-time metric should be such that there is a zero-length geodesic connecting the two space-time points.

- If each satellite has a clock the metric has to be such that the integral of the $\sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$ along each trajectory should correspond to the proper time as given by the clock.

- If the satellites have an accelerometer, the metric has to be such that the computed acceleration (computed via the connection) is close to the observed one.

- If the satellites have a gradiometer, the metric has to be such that the computed tidal accelerations (computed via the Riemann) are close to the observed ones.

- The satellites may have gyroscopes, this providing further information on the space-time connection.

First Constraint on the Metric

In the 'light-coordinates' $\{\tau^\alpha\}$ being used, the contravariant components of the metric must have zeros on the diagonal,

$$\{g^{\alpha\beta}\} = \begin{pmatrix} 0 & g^{12} & g^{13} & g^{14} \\ g^{12} & 0 & g^{23} & g^{24} \\ g^{13} & g^{23} & 0 & g^{34} \\ g^{14} & g^{24} & g^{34} & 0 \end{pmatrix},$$

so the basic unknowns of the problem are the six quantities

$$\{g^{12}, g^{13}, g^{14}, g^{23}, g^{24}, g^{34}\}.$$

This constraint is imposed exactly, by just expressing all the relations of the theory in terms of these six quantities.

The covariant components of the covariant metric are *not* zero.

Second Constraint (Einstein Equation)

The Einstein equation is

$$T_{\alpha\beta} = \frac{1}{\chi} E_{\alpha\beta} \quad ,$$

where $t_{\alpha\beta}$ is the stress-energy tensor, and where $E_{\alpha\beta}$ is the Einstein tensor,

$$E_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \quad ,$$

and where

$$\begin{aligned}\Gamma^{\alpha}{}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\sigma} (\partial_{\beta} g_{\gamma\sigma} + \partial_{\gamma} g_{\beta\sigma} - \partial_{\sigma} g_{\beta\gamma}) \\ R^{\alpha}{}_{\beta\gamma\delta} &= \partial_{\gamma} \Gamma^{\alpha}{}_{\delta\beta} - \partial_{\delta} \Gamma^{\alpha}{}_{\gamma\beta} + \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\delta\beta} - \Gamma^{\alpha}{}_{\mu\delta} \Gamma^{\mu}{}_{\gamma\beta} \\ R_{\alpha\beta} &= R^{\gamma}{}_{\alpha\gamma\beta} \quad .\end{aligned}$$

Third Constraint (Smoothness)

We wish that our final estimation of the metric, \mathbf{g} , is close to the a priori estimation $\mathbf{g}_{\text{prior}}$. More precisely, letting \mathbf{C}_g be a suitably chosen (smoothing) covariance operator, we impose that the least-squares norm

$$\| \mathbf{g} - \mathbf{g}_{\text{prior}} \|_{\mathbf{C}_g}^2 \equiv \langle \mathbf{C}_g^{-1} (\mathbf{g} - \mathbf{g}_{\text{prior}}) , (\mathbf{g} - \mathbf{g}_{\text{prior}}) \rangle$$

is small.

In a more advanced state of the theory, we should introduce the logarithm of the metric, and base the minimization criterion on the difference of logarithmic metrics.

Proper Time Data

At any point along the trajectory of the satellite (whose proper time is σ) we must have $g_{\alpha\beta} d\tau^\alpha d\tau^\beta = d\sigma^2$, or, introducing the four-velocity

$$g_{\alpha\beta} u^\alpha u^\beta = 1$$

Arrival Time Data

Our data set here consists on a set of N values

$$\{ \sigma_{\text{obs}}^i \} \quad ; \quad i = 1, 2, \dots, N \quad .$$

We denote by

$$\mathbf{g} \quad \mapsto \quad \sigma_{\text{computed}} = \sigma(\mathbf{g})$$

the operator that to every conceivable space-time metric field \mathbf{g} associates the computed data. From an algorithmic point of view, for a given \mathbf{g} the computation of $\sigma(\mathbf{g})$ involves taking one by one all the trajects between a source and a receiver, and for each of the trajects compute an arrival time.

Accelerometer Data

The acceleration along a trajectory is

$$a^{\alpha} = u^{\beta} \frac{\partial u^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma} = \frac{du^{\alpha}}{d\sigma} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma} \quad ,$$

where σ is the proper time along the trajectory. The measure of the acceleration provides information on the connection.

Gyroscope Data

If a gyroscope follows a trajectory $x^\alpha = x^\alpha(\sigma)$, whose velocity is u^α and whose acceleration is a^α , the evolution of the spin vector s^α along the trajectory is described by the Fermi-Walker transport:

$$\frac{Ds^\alpha}{d\sigma} \equiv \frac{ds^\alpha}{d\sigma} + \Gamma^\alpha_{\beta\gamma} u^\beta s^\gamma = s_\beta (a^\beta u^\alpha - a^\alpha u^\beta) \quad .$$

Consider that our data is

$$\pi^\alpha = \frac{ds^\alpha}{d\sigma} \quad .$$

Then we have

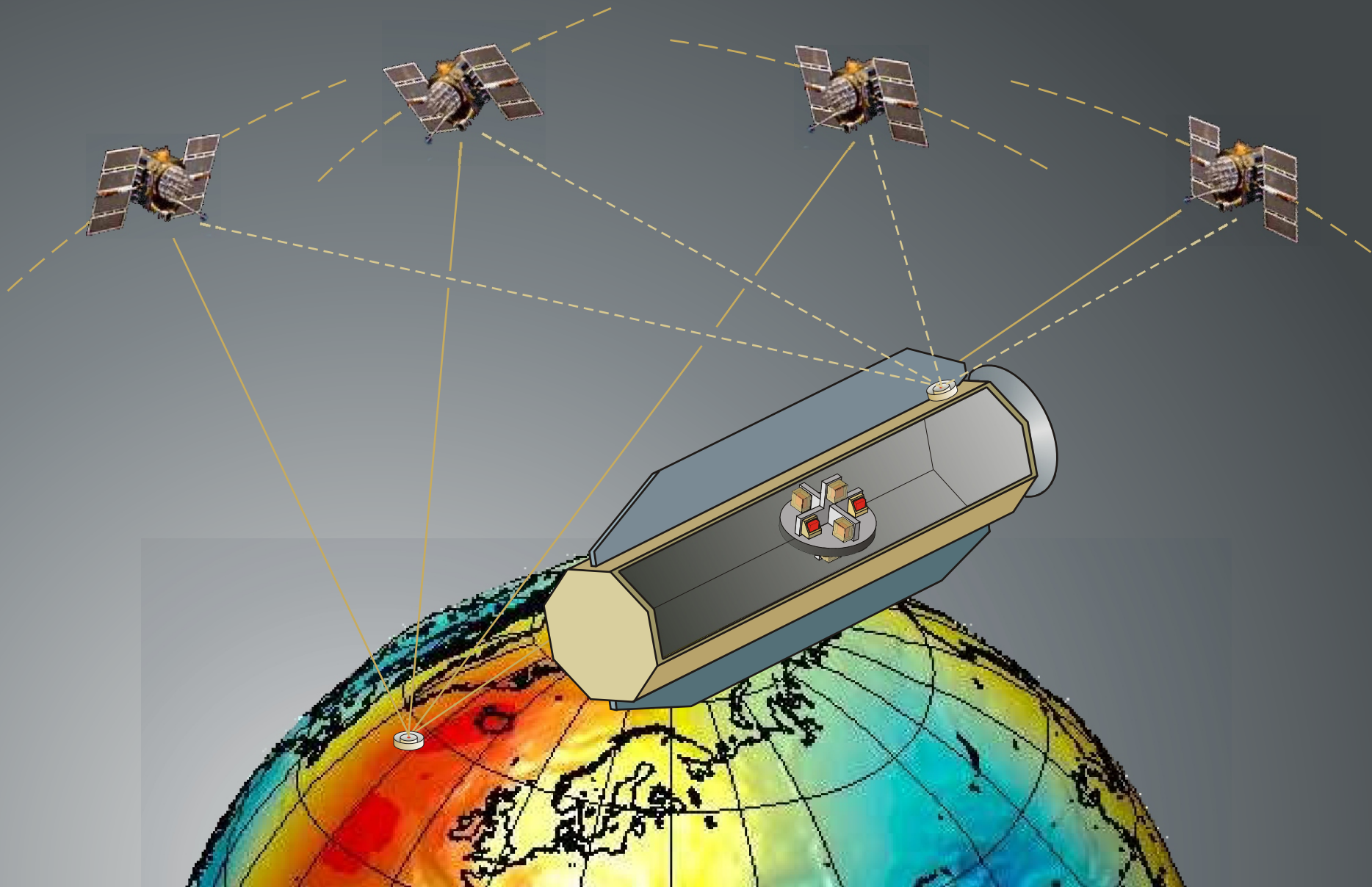
$$\pi^\alpha = s_\beta (a^\beta u^\alpha - a^\alpha u^\beta) - \Gamma^\alpha_{\beta\gamma} u^\beta s^\gamma \quad .$$

Gravity Missions

- The LAGEOS (LAsEr GEOdynamics Satellites) are passive spherical bodies covered with retroreflectors.
- The CHAMP (CHAllenging Minisatellite Payload) satellite is equipped with an accelerometer.
- The GRACE (GRAvity recovery and Climate Experiment) consists in two satellites with accelerometers and measure of their mutual distance with an accuracy of a few microns.
- The GOCE (Gravity Field and Steady-State Ocean Circulation Explorer) satellite will consist in a three axis *gradiometer*.
- and Gravity Probe B...
- and LISA (Laser Interferometer Space Antenna)...

GOCE Gradiometer

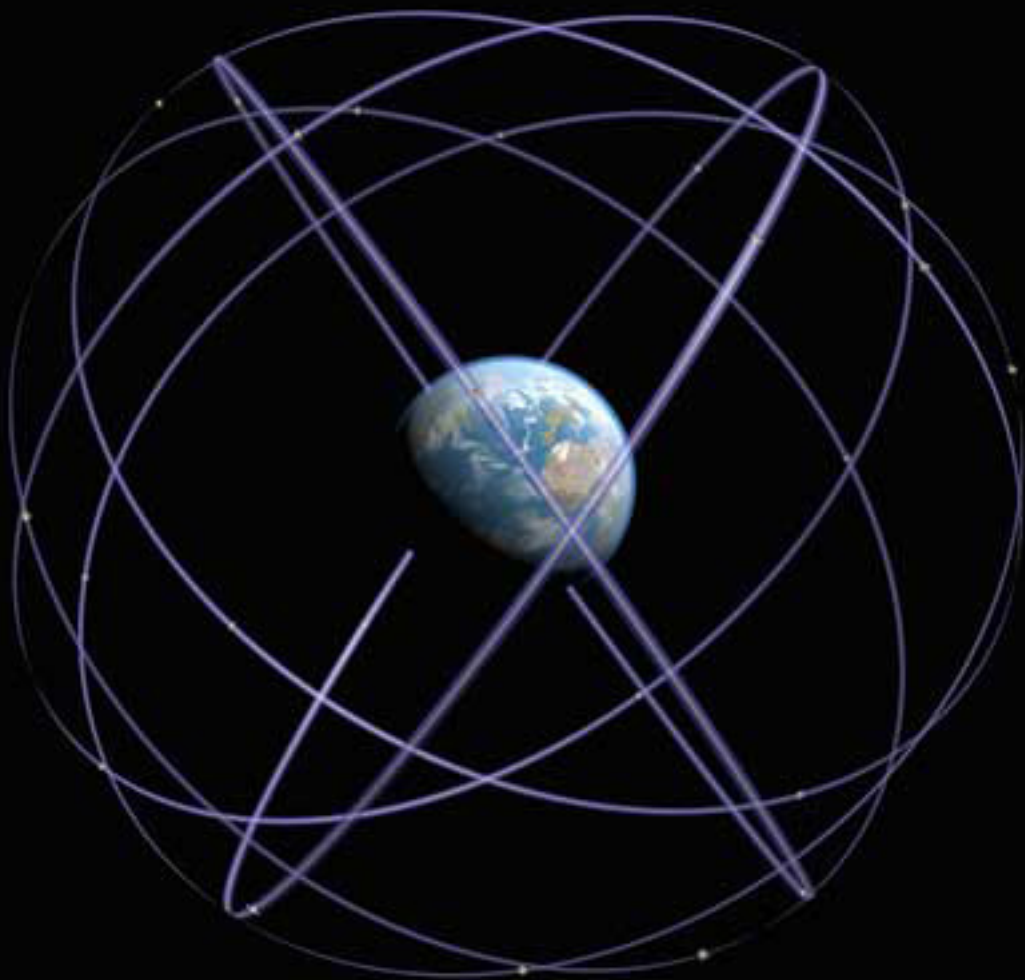
GOCE will employ a **three-axis electrostatic gravity gradiometer** that will allow gravity gradients to be measured in all spatial directions. The measured signal is the difference in gravitational acceleration at the test-mass location inside the spacecraft caused by gravity anomalies from attracting masses of the Earth. Exploiting these differential measurements for all three spatial axes has the advantage that all disturbing forces acting uniformly on the spacecraft (i.e. drag, thrusters resulting in linear or angular accelerations) can be compensated for. The length of the baseline for an accelerometer pair is 50 cm.



Gradiometer Data

At first order in δv^α , $\delta a^\alpha = R^\alpha_{\mu\nu\rho} u^\mu u^\rho \delta v^\nu$. As the three vectors a^α , u^α , and δv^α are known, we obtain information on the components of the Riemann tensor. We drop the δ for the vector δv^α , and we write ω^α instead of δa^α :

$$\omega^\alpha = R^\alpha_{\mu\nu\rho} u^\mu u^\rho v^\nu \quad .$$



The 12 Causal Classes of Newtonian Frames

	teee	ttee	ttte	tttt
eeee	TEEEEE EEEEEE	TEEEEE EEEEEE	TEEEEE EEEEEE	TEEEEE EEEEEE
teee	TTTEEE	TTTEEE	TTTEEE	TTTEEE

t , e

t , e

T , E

are respectively
time-like , space-like
coordinate

hypersurfaces or covectors

lines or tangent vectors

surfaces or planes

The 6 surfaces or planes of every class $X_1X_2X_3X_4X_5X_6$ are generated restectively by the 4 hypersurfaces or covectors x_1, x_2, x_3, x_4 in the following order

$X_1 = x_1 \wedge x_2$
 $X_2 = x_1 \wedge x_3$
 $X_3 = x_1 \wedge x_4$

$X_4 = x_2 \wedge x_3$
 $X_5 = x_2 \wedge x_4$
 $X_6 = x_3 \wedge x_4$

The 199 Causal Classes of Space-time Frames

	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tl1l	tt1l	ttt1	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLLEE TLLLEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLLE TTTTEE LLLLLL TLLLLL TTTTLE TTTTLE TTTTLE TTTTTE LLLLLL TLLLLL TLLLLL TTTTLL TTTTLL TTTTTL TTTTTT	EEEEEE LEEEEE EEELEE TEEEEE LELEEE LEELEE EELLEEE TLEEEE TEEELE EETLEE TEEEEE LLLLEE LLELEE LELLEE TLEEEE TLELEE TELLEE LETLEE TLEEEE TTELEE TETLEE TTTEEE LLLLEE TLLLEE LLTLEE TLLLEE TLTLEE TTTLEE		EEEEEE LEEEEE TEEEEEE LLEEEE TLEEEE TTEEEE LLLEEE EELLEEE ELELEE TLEEEE TLEEEE TTTEEE	EEEEEE LEEEEE EEEEEE EELLEEE ELELEE TLEEEE LLELEE LELLEE TLELEE TELLEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE	EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE EEEEEE
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ttee	TTTTTE TTTTTL TTTTTT															
llle	TTLTLL TTTTLL TTTTTL TTTTTT															
tlle	TTTTLL TTTTTL TTTTTT															
ttle	TTTTTL TTTTTT															
ttte	TTTTTT															
llll	TTTTTT															
tl1l	TTTTTT															
tt1l	TTTTTT															
ttt1	TTTTTT															
tttt	TTTTTT															

t, l, e
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are respectively **time-like** , **light-like** , **space-like** coordinate

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 $X_5 = x_2 \wedge x_4$

$X_3 = x_1 \wedge x_4$
 $X_6 = x_3 \wedge x_4$

Pulsar 1

$(0,0)$

Earth

spacecraft

Pluto

$(3,5)$

Pulsar 2

