# Penn State Astrostatistics MCMC tutorial

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## Bayesian change point model with Gamma hyperpriors

Consider the following hierarchical changepoint model for the number of occurrences $Y_i$ of some event during time interval $i$ with change point $k$.

$$\begin{aligned} Y_i|k,\theta,\lambda &\sim \text{Poisson}(\theta) \text{ for } i = 1,\dots,k \\ Y_i|k,\theta,\lambda &\sim \text{Poisson}(\lambda) \text{ for } i = k+1,\dots,n \end{aligned}$$

Assume the following prior distributions:

$$\begin{aligned} \theta|b_1 &\sim \text{Gamma}(0.5, b_1) \qquad (\text{pdf}=g_1(\theta|b_1)) \\ \lambda|b_2 &\sim \text{Gamma}(0.5, b_2) \qquad (\text{pdf}=g_2(\lambda|b_2)) \\ b_1 &\sim \text{Gamma}(c_1, d_1) \qquad (\text{pdf}=h_1(b_1)) \\ b_2 &\sim \text{Gamma}(c_2, d_2) \qquad (\text{pdf}=h_2(b_2)) \\ k &\sim \text{Uniform}(1,\dots,n) \qquad (\text{pmf} = u(k)) \end{aligned}$$

where $c_1 = c_2 = 0.01$ and $d_1 = d_2 = 100$, $k, \theta, \lambda$ are conditionally independent and $b_1, b_2$ are independent.
Assume the Gamma density parameterization $\text{Gamma}(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

Inference for this model is therefore based on the 5-dimensional **posterior** distribution $f(k,\theta,\lambda,b_1,b_2|\mathbf{Y})$ where $\mathbf{Y}=(Y_1,\dots,Y_n)$. The posterior distribution is obtained *upto a constant* by taking the product of all the conditional distributions. Thus we have

$$\begin{aligned} f(k,\theta,\lambda,b_1,b_2|\mathbf{Y}) &\propto \prod_{i=1}^{k} f_1(Y_i|\theta,\lambda,k) \prod_{i=k+1}^{n} f_2(Y_i|\theta,\lambda,k) \\ &\times g_1(\theta|b_1) g_2(\lambda|b_2) h_1(b_1) h_2(b_2) u(k) \\ &= \prod_{i=1}^{k} \frac{\theta^{Y_i} e^{-\theta}}{Y_i!} \prod_{i=k+1}^{n} \frac{\lambda^{Y_i} e^{-\lambda}}{Y_i!} \\ &\times \frac{1}{\Gamma(0.5) b_1^{0.5}} \theta^{-0.5} e^{-\theta/b_1} \times \frac{1}{\Gamma(0.5) b_2^{0.5}} \lambda^{-0.5} e^{-\lambda/b_2} \\ &\times \frac{1}{\Gamma(c_1) d_1^{c_1}} b_1^{c_1-1} e^{-b_1/d_1} \frac{1}{\Gamma(c_2) d_2^{c_2}} b_2^{c_2-1} e^{-b_2/d_2} \times \frac{1}{n} \end{aligned}$$

If we are able to draw samples from this distribution, we can answer questions of interest.