Aspects of 2-D MHD Flows and Astrophysical Applications

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- Properties of 2-D MHD flows
- Formulation and Numerics
- Application to different situations with applications to Active Galactic Nuclei (AGN), Young Stellar Objects (YSO), Cataclysmic Variables (CV), and X-ray Binaries (XRB)
- Working towards an Electrodynamical jet emerging from a disk dynamo

Properties of steady axisymmetric flow

It is instructive to examine properties of MHD flow under perfect conductivity and axisymmetry in some detail. The basic conservation laws were worked out by Chandrasekhar (1956) and redone by Mestel (1961) and applied to rotating stars; an algebraic variation of the latter calculation follows.

- Under the conditions assumed here, we can write the induction equation as

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = 0,$$

  where by axisymmetry one can write the velocity and magnetic fields as

$$\mathbf{B} = \mathbf{B}_p + \mathbf{B}_\phi \qquad \text{and} \qquad \mathbf{v} = \mathbf{v}_p + \mathbf{v}_\phi,$$

  in the usual cylindrical coordinate system; the toroidal velocity is $\mathbf{v}_\phi = \boldsymbol{\omega} r$.

- Since the toroidal term, $\mathbf{v}_p \times \mathbf{B}_p$, cannot be a gradient of a potential, the poloidal part of the flux freezing condition implies that

$$\mathbf{v}_p = \frac{\kappa(\psi)}{4\pi\rho}\mathbf{B}_p,$$

  where κ is a constant on each magnetic surface, ψ.

- The toroidal part of the condition in combination with the above result, and the definitions of the fields, yields

$$\begin{aligned} \nabla \times (\mathbf{v}_\phi \times \mathbf{B}_p + \mathbf{v}_p \times \mathbf{B}_\phi) &= 0 \\ (\mathbf{B}_p \cdot \nabla)\mathbf{v}_\phi - (\mathbf{v}_\phi \cdot \nabla)\mathbf{B}_p - (\mathbf{B}_p \cdot \nabla)(\frac{\kappa(\psi)}{4\pi\rho}\mathbf{B}_\phi) + (\mathbf{B}_\phi \cdot \nabla)(\frac{\kappa(\psi)}{4\pi\rho}\mathbf{B}_p) &= 0 \\ (\mathbf{B}_p \cdot \nabla - \frac{B_r}{r})(\frac{\kappa(\psi)}{4\pi\rho}\mathbf{B}_\phi - \mathbf{v}_\phi) &= 0 \end{aligned}$$

$$
\begin{aligned}
(\mathbf{B}_p \cdot \nabla)(\frac{\kappa(\psi)}{4\pi\rho r}\mathbf{B}_\phi - \boldsymbol{\omega}) &= 0 \\
\frac{\kappa(\psi)}{4\pi\rho r}B_\phi - \omega &= \omega_f(\psi),
\end{aligned}
$$

where $\omega_f(\psi)$ is the angular speed of the magnetic surface and is equal to the Keplerian value at the footpoint; this result may be termed as the slippage condition, since it describes the slip between the fluid particle and the surface.

- Next, we write the Euler's equation for a magnetized, warm gas in a gravitational potential, φ_g, as

$$
\nabla\left[(v_p^2 + \omega^2 r^2)/2 + \varphi_g + H\right] = \frac{1}{4\pi\rho}(\nabla \times \mathbf{B}) \times \mathbf{B} - (\nabla \times \mathbf{v}) \times \mathbf{v},
$$

where H is the specific enthalpy of the gas. The toroidal component of this equation, to which the gradient term does not contribute, is

$$
\begin{aligned}
(\mathbf{B}_p \times \nabla \times \mathbf{B}_\phi)/4\pi\rho &= \mathbf{v}_p \times \nabla \times \mathbf{v}_\phi \\
\nabla \times \mathbf{B}_\phi - \kappa(\psi)\nabla \times \mathbf{v}_\phi &\propto \mathbf{B}_p \\
\hat{P}(rB_\phi) - \kappa(\psi)\hat{P}(r^2\omega) &\propto \hat{P}(\psi) \\
r^2\omega - \frac{rB_\phi}{\kappa(\psi)} &= \ell(\psi) \equiv \omega_f(\psi) r_A^2(\psi),
\end{aligned}
$$

and the integral of motion that results is the conservation of total angular momentum of the system, $\ell(\psi)$, with its lever arm defined as the Alfvén radius, r_A, containing both the gas and field contributions.

- The flow energetics involving the Bernoulli and Grad-Shafranov equations (to be derived below) were used to describe the plasma confinement in tokomaks (e.g. Shafranov 1965) and in the vicinity of pulsars (e.g. Michel 1973). The Bernoulli equation (see for e.g., Chandrasekhar 1961) can be written as

$$1/2(v_p^2+\omega^2r^2)+\varphi_g+H+\omega_f(\omega_f r_A^2-\omega r^2)=E(\psi),$$

where the last term on the left is the work done by the field on the gas, while the specific energy, $E(\psi)$, is constant along streamlines. This may also be obtained by taking the component along the magnetic surface. It can be seen from the Bernoulli equation that the terminal speed of the gas will be

$$v_\infty(\psi)\sim\omega_f r_A,$$

if gravity is neglected at large distances from the source.

- Taking the gradient of ()

$$\nabla E=d_\psi E\nabla\psi=\frac{1}{4\pi\rho}(\nabla\times\mathbf{B})\times\mathbf{B}-(\nabla\times\mathbf{v})\times\mathbf{v}+\nabla[\omega_f(\omega_f r_A^2-\omega r^2)],$$

we find it is perpendicular to the magnetic surface. The various terms of the RHS are

$$(\nabla\times\mathbf{v}_\phi)\times\mathbf{v}_\phi=\hat{P}(r\omega)\times r\omega\hat{\phi}=-\omega\nabla(r^2\omega),$$

and

$$(\nabla\times\mathbf{B}_p)\times\mathbf{B}_p=\mathbf{B}_p\times(\hat{\phi}\frac{\Lambda\psi}{r})=-\frac{1}{r^2}(\nabla\psi)(\Lambda\psi),$$

- The following identities are useful:

$$\hat{P}=-\frac{\hat{r}}{r}\partial_z+\frac{\hat{z}}{r}\partial_r\ \text{ and }\ \Lambda=r\partial_r\frac{1}{r}\partial_r+\partial_z^2,$$

where the partial derivatives have been abbreviated. We point out that the r and z eigenfunctions of Λ are

$$\Lambda_r=r\ J_1(k_r r)\text{ and }\Lambda_z=\exp(k_z z).$$

Now,

$$[\nabla^2 \hat{P}]_z = (\frac{1}{r}\partial_r r \partial_r + \partial_z^2)(\frac{1}{r}\partial_r) = (\frac{1}{r}\partial_r)(r\partial_r \frac{1}{r}\partial_r + \partial_z^2) = \hat{P}_z \Lambda,$$

for the z component and for the r component, we find that

$$[\nabla^2 \hat{P}]_r = (\partial_r \frac{1}{r}\partial_r r + \partial_z^2)(-\frac{1}{r}\partial_z) = -\partial_z \partial_r \frac{1}{r}\partial_r - \frac{1}{r}\partial_z \partial_z^2 = \hat{P}_r \Lambda.$$

Similarly one can show

$$\nabla \times \hat{P} = -\hat{\phi}\,\frac{\Lambda}{r} \;\text{ and }\; \hat{P}(r\Omega) = \nabla \times (\Omega\hat{\phi})$$

- The term

$$\begin{aligned}(\nabla \times \mathbf{v}_p) \times \mathbf{v}_p &= (\frac{\kappa}{4\pi\rho})^2[-\frac{1}{r^2}\nabla\psi\Lambda\psi] + \frac{\kappa}{4\pi\rho}B_p^2\nabla_p(\frac{\kappa}{4\pi\rho}) \\ &= \frac{\nabla\psi}{4\pi\rho r^2}\left[-(\frac{\kappa}{4\pi\rho})^2\Lambda\psi + \frac{1}{2}\nabla\psi\cdot(\frac{\kappa}{4\pi\rho})^2\right] \\ &= \frac{\nabla\psi}{4\pi r^2}[\frac{\rho_A}{\rho^2}\Lambda\psi + \frac{1}{2}\nabla(\frac{\rho_A}{\rho^2})\cdot\nabla\psi]\end{aligned}$$

and the hoop stress

$$(\nabla \times \mathbf{B}_\phi) \times \mathbf{B}_\phi = \hat{P}(rB_\phi) \times B_\phi\hat{\phi} = -\frac{1}{2r^2}\nabla(rB_\phi)^2.$$

The final term in () works out to be

$$\nabla(\omega_f^2 r^2 \rho g/\rho_A) + \nabla(\omega_f^2 r_A^2),$$

where $\rho_A(\psi) \equiv \kappa^2/4\pi$ is the density at the Alfvén point (a ring on the magnetic surface where the poloidal velocity reaches Alfvén speed) and

$$g = \frac{\omega_f - \omega}{\omega_f} = \frac{r_A^2 - r^2}{r^2}\frac{\rho_A}{\rho - \rho_A},$$

measures the angular velocity of the gas with respect to the fields; so that $0 < g < 1$ as the gas asymptotically loses its angular momentum to the fields. Using the above results it can be shown that

$$\omega_f r_A^2 - \omega r^2 = \omega_f r^2 g\rho/\rho_A.$$

- En route, it is found that

$$-\frac{\nabla(rB_\phi)^2}{8\pi r^2} - g\omega_f \nabla(\omega r^2) = g\omega_f[-\frac{g\omega_f r^2}{\rho_A^2}\partial_\psi \rho_A - \nabla(\omega r_A^2)]\nabla\psi.$$

The last two terms along with the hoop stress provide the effects of collimation and hence, after collecting all terms, the Trans-Field equation can written as

$$\frac{\rho_A - \rho}{\rho}\Lambda\psi + \frac{1}{2}\rho\nabla\psi \cdot \nabla(\frac{\rho_A}{\rho^2}) = 4\pi\rho r^2[d_\psi E + c(\psi, g; r)],$$

where, after some additional effort, the collimation term is found to be

$$c(\psi, g; r) = g\omega_f[d_\psi(\omega_f r_A^2) + g\omega_f r^2 \frac{\rho}{2\rho_A^2} d_\psi \rho_A] + (1-g)\omega_f r^2 d_\psi(\omega_f) - d_\psi(\omega_f^2 r_A^2).$$

In summary,

- The Trans-Field equation also describes the decollimating effects of the centrifugal motion pushing against the magnetic surface. The full set of MHD wind equations (Bernoulli and the Trans-Field) given above, are written in a general form including gravity, internal energy of the gas, field energy, and rotation, and contain the two functions $\psi(r, z)$ and $\rho(r, z)$, and four invariant functionals of ψ, namely, ρ_A, r_A, ω_f and E.

- At this point, the various models diverge in philosophy and application; a broad classification can be made into Electrodynamical and MHD models.
- In MHD models, inertial effects (terms that involve ρ_A) of the gas are on an equal footing with the electromagnetic terms, while in Electrodynamical models, a relativistic force-free condition is imposed in the magnetosphere of the accretion disk and the electromagnetic forces are assumed to dominate over inertial forces such as gravity and thermal pressure.

Critical surfaces and Technical issues

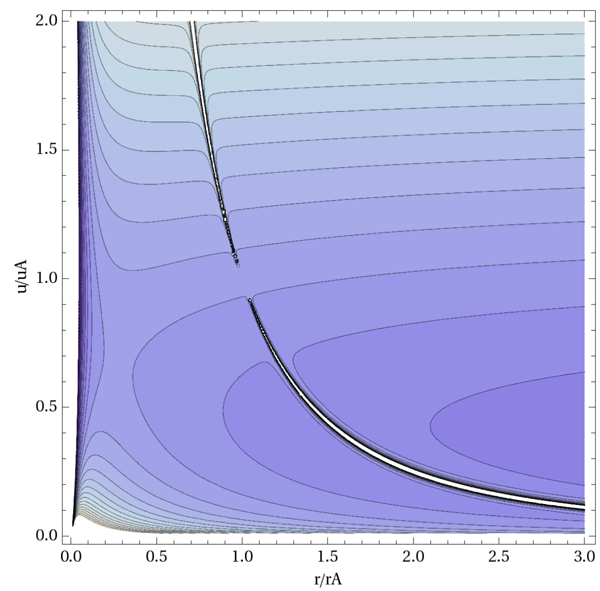

Figure 1: Flow lines passing through critical points in a 1D system

- For cold MHD flows, where only gravity and centrifugal forces matter, the slow magnetosonic point is reached when the effective gravity vanishes (which presumably occurs not far from the disk surface) and the gas will be centrifugally accelerated beyond this point; the thermal pressure merely determines the initial speed at the slow point and is irrelevant to the rest of the flow. Subsequently, at the Alfvén point defined earlier, the poloidal Alfvén speed is reached.

- The singularity at this Alfvén surface (a locus of the Alfvén points) can be partially resolved by demanding that the density, $\rho = \rho_A$, so that $m \equiv v_p(4\pi\rho)^{1/2}/B_p = 1$ defines the critical surface. In the super-Alfveńic regime the gas density is smaller than ρ_A.
- Another regime is the fast magnetosonic flow which is characterized by

  $$n \equiv v_p(4\pi\rho)^{1/2}/(B_p^2 + B_\phi^2)^{1/2}$$

  and the fast magnetosonic surface is defined by $n = 1$.
- The slow point is passed when the gas exits the disk atmosphere, and the demand that the flow must pass smoothly through the two other critical surfaces yields two conditions. This, in addition to a given distribution of ω_f and E on the disk surface, completely defines the problem; the Grad-Shafranov and Bernoulli equations can then be solved for ρ and ψ. In particular, the conditions are obtained by taking a determinant of the two equations expanded to first order at the surfaces, i.e., the definition of critical surfaces yielded by both the equations should be the same. But, the full cold non-linear MHD flow problem with the four constraints is complicated.
- We can simplify the issues by calculating the solutions in the asymptotic fast magnetosonic linear regime, $n = 1$, in addition to making assumptions of self-similarity (which implies that the Alfvén surface is a cone) and it can be shown that the solutions collimate to cylinders in the asymptotic regime
- We provide a class of self-similar solutions with a power law form for ψ in the asymptotic regime. The condition for collimation in their formulation requires that $B_p \sim r^{\mathrm{x}}$, where x < -1.

Astrophysical Applications

- The problem involves poloidal acceleration of the plasma using a hydromagnetic scheme outlined in the introduction. The main intent here is to match dynamo solutions emerging from the disk atmosphere to the magnetic structure of the wind. A complete solution should provide the run of density, velocity and energy along the stream lines. The usual simplification of axisymmetry is made and a steady state is assumed.

Table 1: Properties of accretion systems

Type	Central object	$\mathcal{M}/\mathcal{M}_\odot$	$\dot{m}/\mathcal{M}_\odot \text{yr}^{-1}$	r_{in} (cm)	r_{out}(cm)	B(r_{in}) (G)	v_{jet} (cm/s)
AGN	Black hole(BH)	$10^6 - 10^9$	0.1-10	$10^{12} - 10^{14}$	10^{18}	10^3	$\sim$ c
XRB	BH/Neutron star	1-10	$10^{-10} - 10^{-8}$	$10^6/10^8$	10^{12}	$10^6/10^8$	$\sim 10^{10}$
CV	White dwarf	< 1	$10^{-10} - 10^{-8}$	10^9	10^{11}	10^3	?
YSO	Protostar	1	$10^{-7} - 10^{-4}$	10^{12}	10^{17}	100	10^7

- A distinction is made between the Electrodynamic force-free models where the inertia is ignored, and those that include inertial forces in the treatment.
- A further simplification involves self-similarity, i.e., the properties of the solution are independent of the location, arising from an expectation that the extraction of angular momentum ($\propto r^{1/2}$) scales similarly in radius as magnetic torque ($\sim \psi^2/r \sim B^2 r^3$) which demands that the magnetic field scale as $r^{-5/4}$. Since the dynamo fields we have derived are not self-similar, a general treatment is necessary.

An Electrodynamical model

- Here we present the development of the theory based on earlier work involving steady MHD flow and the Grad-Shafranov equation, and formulate the problem with appropriate boundary conditions. Relevance of the solutions to observables such as the terminal speed of the jet, the luminosity or the thrust in the jet, etc., in various objects is also briefly discussed.
- In this model, the generated dynamo field is matched to a relativistic jet emerging from a force-free magnetosphere. The physics governing the magnetosphere is given by the pulsar equation which is derivable from the general form of Trans-field equation. However, it is worthwhile to obtain it from first principles. The conditions in a magnetosphere which is dominated by electromagnetic fields are given by

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e,$$

$$\nabla \times \mathbf{E} = 0,$$

$$\nabla \times \mathbf{B} = (4\pi/c)\mathbf{j_e},$$
$$\mathbf{E} + (1/c)\mathbf{v} \times \mathbf{B} = 0,$$
$$\rho_e \mathbf{E} + (1/c)\mathbf{j_e} \times \mathbf{B} = \mathbf{0}.$$

- Note that the conservation laws, derived earlier, still hold. The perfect conductivity condition provides

$$\mathbf{E} = -(1/c)\mathbf{v} \times \mathbf{B} = -(1/c)[\omega - \kappa(\psi)B_\phi/(4\pi\rho r)]\nabla\psi = -(1/c)\omega_f \nabla\psi$$

where the earlier definitions, and the slippage condition were applied.

- The conservation of angular momentum implies

$$rB_\phi = T(\psi),$$

where the angular momentum of the gas is neglected in comparison with the poloidal current, $T(\psi)$. Next, by taking the component of the relativistic Euler equation parallel to $\nabla\psi$, one obtains

$$\begin{aligned}
[\omega_f \nabla^2\psi - \partial_\psi \omega_f (\nabla\psi)^2]\omega_f/c^2 - [\frac{1}{r^2}](\nabla(T^2)/2 - \nabla\psi\Lambda\psi) &= 0 \\
(1 - [\omega_f r/c]^2)\Lambda\psi - (1/2r^2)[\nabla(\omega_f r^2/c)^2] \cdot \nabla\psi + T\partial_\psi T &= 0,
\end{aligned}$$

- For convenience we define

$$L(r) = (\omega_f r/c)^2 = (r/r_L)^2$$

where r_L is the light cylinder radius (the radius at which the magnetic surface rotational velocity equals the speed of light). Now we make an assumption: the field is taken to be the form

$$T(\psi) = \mu\psi,$$

which is obeyed by non-relativistic force-free fields (note that ψ is to be determined by the pulsar equation below).

- In the current electrodynamical model, we can consider a trial problem to match to the steady dynamo generated structure. With the above ansatz, we can reduce the relativistic Euler equation to the linearized pulsar equation

$$r^2\nabla \cdot \left[\frac{1}{r^2}(1-L(r))\nabla\psi\right] + \mu^2\psi = 0,$$

  which is degenerate at $r = r_L$.

- It is seen in the non-relativistic limit, $r_L \to \infty$ ($L = 0$), that the force-free eigenfunctions $r\ J_1(k_n^1 r)\ \exp(-\sqrt{{k_n^1}^2 - \mu^2} z)$, are recovered. In order to complete the formulation the boundary conditions at the disk-jet interface need to be specified. The approach taken here is to give precedence to the dynamo generated field, i.e., the disk solutions are expanded in terms of eigenfunctions of the jet (pulsar) equation. The solutions are evaluated by the same boundary conditions as before, which is equivalent to demanding an absence of a sheet current on the disk surface, and the solenoidal nature of the field. Hence,

$$[\psi](1) = 0, \quad [\frac{\partial\psi}{\partial z}](1) = 0,$$

  and

$$b_n(1) = \mu a_n(1),$$

  where the brackets indicate continuity.

- In this formulation the entire magnetic topology (disk and jet) is decided by the dynamo number, the wavelength of the external force-free fields, and the spin of the magnetic surface.

Approach to analytical solutions

- The relativistic equation can be written as

$$\left\{ r\partial_r[\frac{(1-L(r))}{r}\partial_r] + (1-L(r))\partial_z^2 + \mu^2 \right\} \psi = 0$$

which can be simplified to a *Sturm-Liouville* type equation in r, as z decouples from the system via $\psi = \exp(iqz)G(r)$.

- The resulting equation has a singularity at r_L and therefore the solutions on either side of r_L should obey

$$[(2/r)\partial_r G = \mu^2 G]_{r=r_L}.$$

- Then one has to connect the solutions which can be evaluated separately, using the above condition. A well known property of the *Sturm-Liouville* equation is that it yields orthonormal eigenfunctions and eigenvalues.

Remarks on observables

The important observable that wind theory can provide is the luminosity of the jet. In MHD models, this is simply

$$L_{jet} = \int \partial_\psi [\dot{M}_w(\psi)] v_\infty^2 \mathrm{d}\psi,$$

where

$$\dot{M}_w(\psi) = \int \rho \mathbf{v}_p \cdot \hat{z}\ 2\pi r \mathrm{dr} = (1/2) \int_0^\psi \kappa(\psi')\mathrm{d}\psi'$$

is the wind mass loss rate within the magnetic surface, ψ, and the previous equation follows from its definition, (), and Stoke's theorem. Similarly the angular momentum carried away by the wind can be estimated and this contributes directly to the accretion in the disk. It is shown that for a finite current on the axis, the asymptotic wind velocity and the luminosity of the jet are in agreement with observed values for both YSOs and AGN.

In the Electrodynamical model considered above, the wind luminosity is calculated from the Poynting vector, $\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$,

$$L_{jet} = \int \mathbf{S} \cdot \hat{z}\ 2\pi r \mathrm{dr} = \frac{\mathrm{c}}{2} \int \mathrm{E_r B_\phi r dr} = -\int \omega_\mathrm{f} \mathrm{T}(\psi)\mathrm{d}\psi,$$

where the E_ϕ term is zero (gradient of an axisymmetric potential). Note that the inertia is ignored compared to the electromagnetic energy. In the formulation above it follows that

$$L_{jet} = -\frac{\mu}{2}\omega_f \psi_j^2,$$

where ψ_j is the flux at the jet radius. It is clear from this expression that a radiative boundary condition has to be applied, in order that the jet carry away electromagnetic energy.

Summary & Conclusions

- A general treatment of axisymmetric MHD flows was considered using a new approach. A by product of the analysis are solutions for inviscid conducting rotating fluids. A set of unique constraints on magnetic stream functions have been found. They could be applied to superconducting and superfluid flows with possible applications to terrestial plasma experiments or to neutron stars interiors.
- A MHD wind passing through 3 critical surfaces has been formulated that *self consistently* matches to equatorial dynamo generated flux. The dynamo flux eigen functions have been *calculated*. Such solutions will be used to produce non-self similar flow geometries.
- There are subtle but essential difference is the wind geometries for AGN, XRB, and YSOs. The estimates for the jet luminosities and terminal velocities are reasonable. The correct calculation of r_A is crucial in determing v_∞.
- It is worth exploring the ED model in the simplest form. Realistic models can be extended from these.