HYDRODYNAMIC STABILITY AND STELLAR OSCILLATIONS

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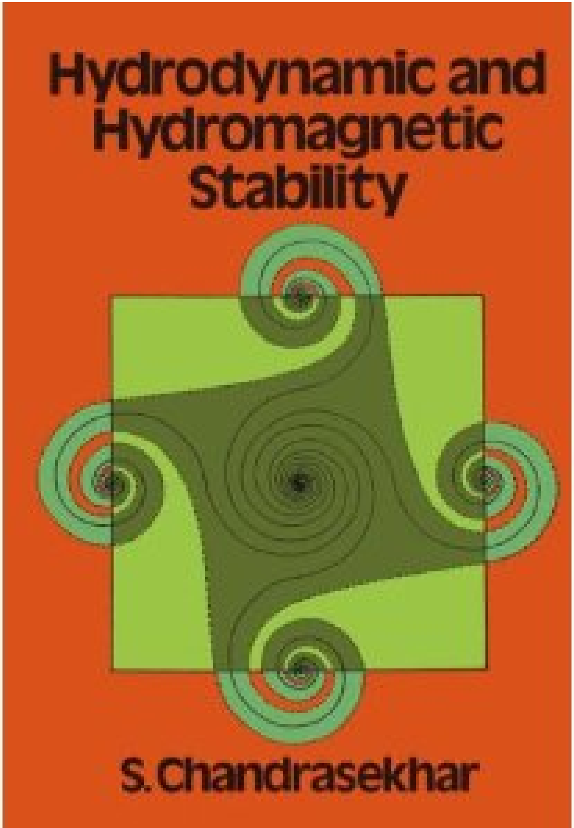
Mumbai


Hydrodynamic and
Hydromagnetic
Stability
S. Chandrasekhar

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A GENERAL VARIATIONAL PRINCIPLE GOVERNING THE RADIAL AND THE NON-RADIAL OSCILLATIONS OF GASEOUS MASSES

S. CHANDRASEKHAR
University of Chicago
Received September 30, 1963

ABSTRACT

In this paper a general variational principle, applicable to radial as well as non-radial oscillations of gaseous masses, is formulated and proved. And it is, further, shown that when the normal modes are analyzed in vector spherical harmonics, the variational principle requires that the square of the characteristic frequency of oscillation, σ^2, belonging to a particular spherical harmonic, is stationary with respect to simultaneous variations of two independent radial functions. A consequence of this result is that σ^2 (belonging to a particular harmonic) emerges as a characteristic root of a 2×2 matrix.

Two simple illustrations of the variational principle are given.

I. INTRODUCTION

A general variational principle applicable to radial as well as non-radial oscillations of gaseous masses has been formulated recently (Chandrasekhar 1963). In this paper the principle will be cast in a form in which it can be applied, directly, to determining the characteristic frequencies of the normal modes of oscillation belonging to the different spherical harmonics. For the sake of completeness, the principle will be rederived with, however, some amplifications.

II. THE VARIATIONAL PRINCIPLE

Consider a spherically symmetric configuration in equilibrium under its own gravitation and governed by the equations

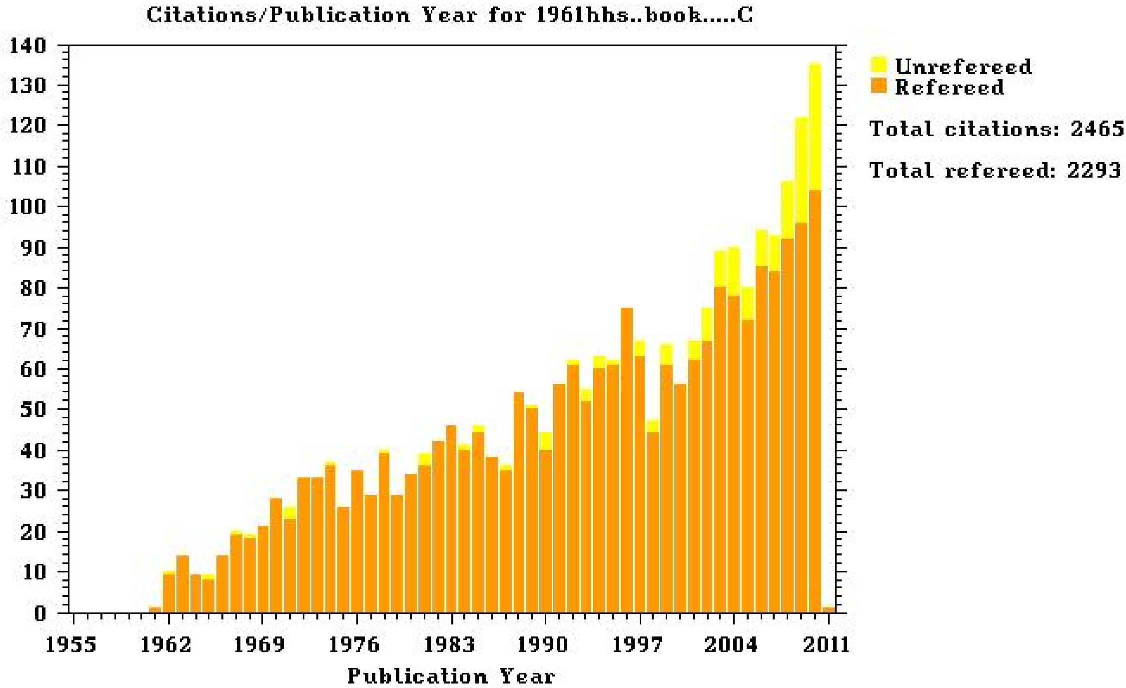
$$\frac{dp}{dr} = \rho \frac{d\mathfrak{B}}{dr} = -\frac{GM(r)}{r^2}\rho \,, \tag{1}$$

and

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\mathfrak{B}}{dr}\right) = -4\pi G\rho \,, \tag{2}$$

ApJ 139, 664

Monograph	Year	Pages	Citations
An Introduction to the Study of Stellar Structure	1939	508	1024
Principles of Stellar Dynamics	1943	313	701
Radiative Transfer	1950	393	1766
Hydrodynamic and Hydromagnetic Stability	1961	652	2465
Ellipsoidal Figures of Equilibrium	1969	252	862
The Mathematical Theory of Black Holes	1983	646	1121


Citations/Publication Year for 1961hhs..book.....C
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2011
Publication Year
Unrefereed
Refereed
Total citations: 2465
Total refereed: 2293

Hydrodynamic and Hydromagnetic Stability

1. Basic concepts

2–6. The thermal instability of a layer of fluid heated from below

7–9. The stability of Couette flow

10. The stability of superposed fluids: The Rayleigh–Taylor instability

11. The stability of superposed fluids: The Kelvin–Helmholtz instability

12. The stability of jets and cylinders

13. Gravitational equilibrium and gravitational instability

14. A general variational principle

The Benard Problem

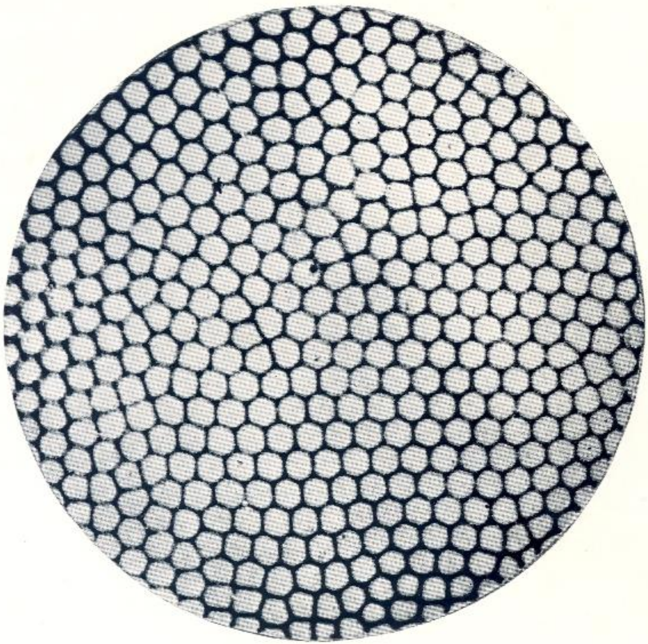


Fig. 1. Bénard cells in spermaceti. A reproduction of one of Bénard's original photographs.

- Boussinesq approximation: Fluid is treated as incompressible and variations in density are ignored except in the gravitational force (buoyancy) term, which is written as

$$\delta\rho\mathbf{g} = -\alpha\rho_0\delta T\mathbf{g},$$

where α is the coefficient of volume expansion and

$$\rho = \rho_0\left[1 - \alpha(T - T_0)\right]$$

.

- This approximation filters out the acoustic modes of oscillation and is applicable for laboratory experiments with liquids.

- With these approximations the linearised perturbation equations have constant coefficients and the normal modes have solution of the form

$$\exp(i(k_x x + k_y y) + pt), \qquad k = \sqrt{k_x^2 + k_y^2}.$$

- Equations are written in dimensionless form using d the height of fluid layer as the dimension of length and d^2/ν as dimension of time.

- The equation depends on some dimensionless constants, i.e., the Rayleigh number:

$$R = \frac{g\alpha\beta d^4}{\kappa\nu}$$

the Prandtl number $\mathcal{P} = \nu/\kappa$, and the wavenumber $a = kd$ and $\sigma = pd^2/\nu$.

- The equations along with appropriate boundary conditions can be solved to obtain the dispersion relation for $\sigma(R, \mathcal{P}, a)$. Here, the wavenumber a defines the length scale of perturbations and for stability the $\mathrm{Real}(\sigma) < 0$ for all values of a.

- Chandrasekhar has done detailed analysis of the equation for different boundary conditions to analyse stability.

- Since σ is a continuous function of its argument, the transition to instability occurs when $\mathrm{Real}(\sigma) = 0$, which yields two possibilities, $\sigma = 0$ or σ is purely imaginary.

- When σ is real the state of marginal stability is given by $\sigma = 0$ and when that happens it is referred to as principle of the exchange of stability. Chandra has shown that for the non-rotating case this is the route to instability.

- In this case the equation doesn't depend on $\mathcal{P}$ and yields $R(a)$. Minimising this function w.r.t. a gives the critical Rayleigh number, R_c.

- For instability R has to be greater than this critical value, while the value of a which gives minimum R gives the length scale of modes that would be excited when R just exceeds the critical value.

- Although the linear stability analysis doesn't give any information about the shape of convective cellular pattern, Chandra has analysed various possibilities and compared them with experimentally observed patterns.

- The second possibility of purely imaginary σ leads to the possibility of transition to instability through oscillatory modes, which is referred to as overstability. This possibility can arise in the presence of rotation or magnetic field.

- Rotation introduces additional forces and the simplest case arises when the rotation axis is parallel to gravitational force. In presence of rotation the equations involve another dimensionless constant, the Taylor number

$$T = \frac{4\Omega^2 d^4}{\nu^2}$$

- In this case if the onset of instability is through stationary convection ($\sigma = 0$), then the critical Rayleigh number is a function of T and its value is found to increase with T.

- However, there is another possibility that the onset of instability arises through overstability, or oscillatory instability. This is possible when $\mathcal{P} < 1$. Minimum value of R for instability depends on $\mathcal{P}, T$. In case this minimum is less than that for instability through stationary convection, then the transition occurs through overstability.

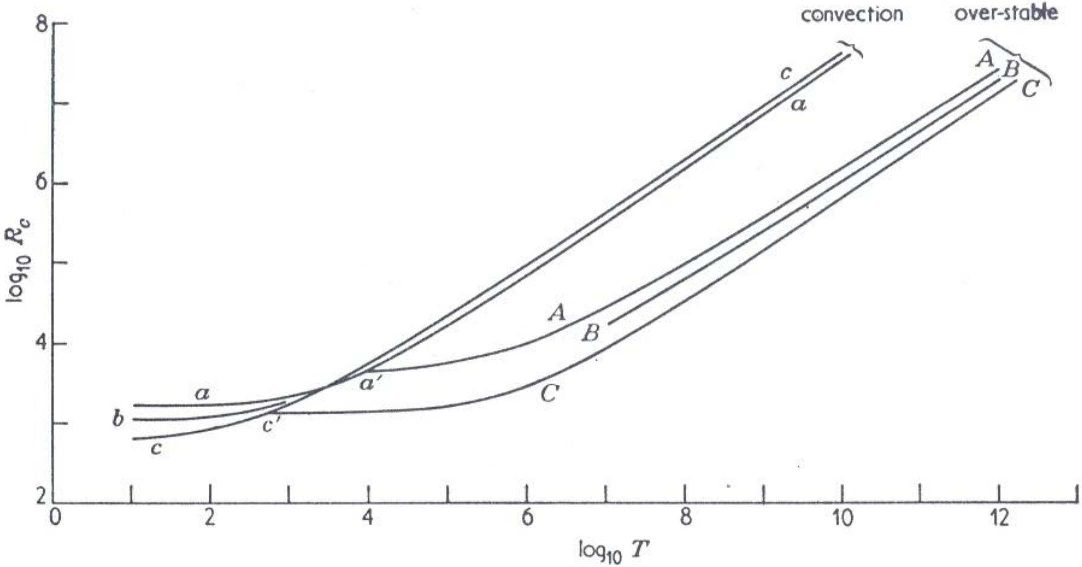


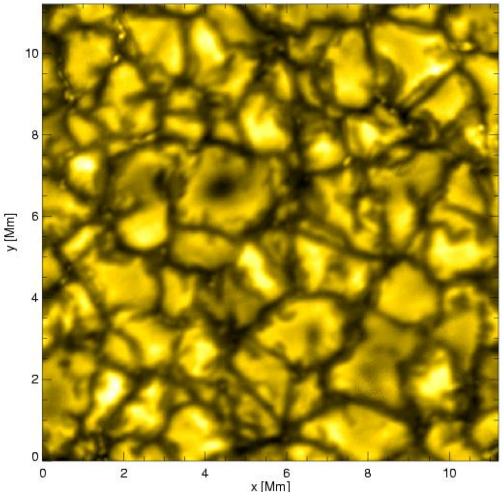
FIG. 21. The (R_c, T)-relations for the three cases (i) both bounding surfaces rigid, (ii) one bounding surface rigid and the other free, and (iii) both bounding surfaces free; the curves labelled aa, b, and cc are the relations for the onset of ordinary cellular convection for the three cases, respectively. The curves labelled $a'AA$, BB, and $c'CC$ are the corresponding relations for the onset of overstability for $\mathfrak{p} = 0{\cdot}025$. At a' (respectively c') we have a change from one type of instability to another as T increases.

- Similar results were obtained with magnetic field

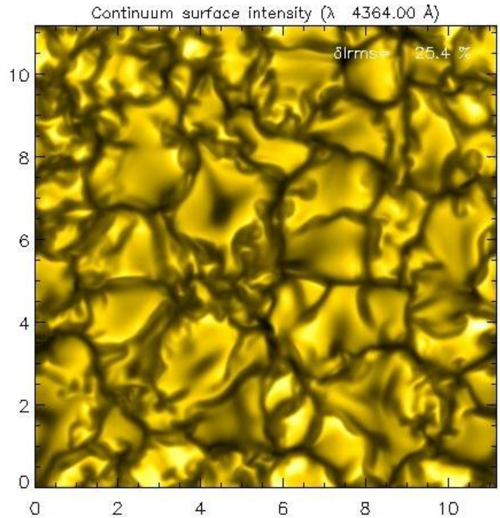
- This work has been extended to nonlinear case where the Rayleigh no. is marginally above the critical value. This can give information about the pattern of convective cells.

- Another extension is to compressible fluid, which is more relevant for stellar convection. In this case the eigenvalue problem has to be solved numerically.

- With enhanced computing power it is now possible to do fully nonlinear calculations for fluid in a box. Although, these calculations cannot yet account for all scales of turbulence in stellar convection zone.



Solar Granulation



Numerical Simulation

Swedish Solar Telescope on La Palma by Carlsson et al.

CO^5BOLD by Matthias Steffen and Bernd Freytag

Some papers citing HHS

- Pattern formation outside of equilibrium
  M. C. Cross and P. C. Hohenberg
  RvMP (1993) 65, 851

- Pattern formation in Benard convection ($R > R_c$) and Couette flow

- Theory of extragalactic radio sources
  M. C. Begelman, R. D. Blandford, M. J. Rees
  RvMP (1984) 56, 255

- Formation of jets and Kelvin-Helmholtz instability

- Instabilities and pattern formation in crystal growth
  J. S. Langer (1980) RvMP 52, 1
- Stability analysis and state of marginal stability

- The dynamical state of the interstellar gas and field
  E. N. Parker (1966) ApJ 145, 811
- Rayleigh-Taylor instability in presence of magnetic field

- Gas dynamics of semidetached binaries

  S. H. Lubow and F. H. Shu (1975) ApJ 198, 383

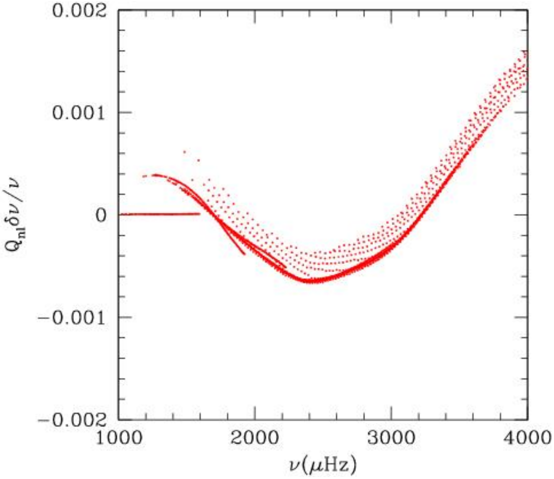
- Kelvin-Helmholtz instability

- Stellar turbulent convection — A new model and application

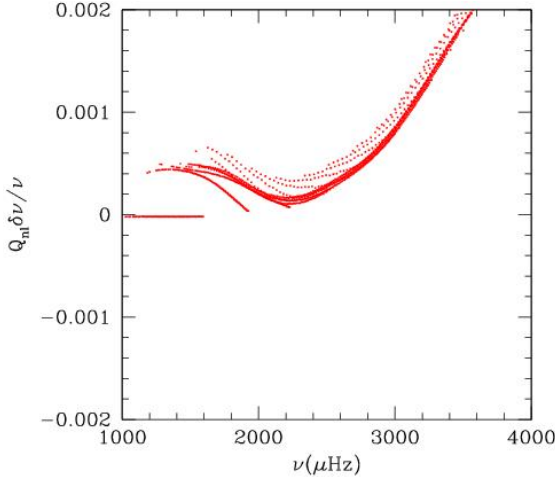
  V. M. Canuto and I. Mazzitelli (1991) ApJ 370, 295

- Calculation of convective flux in stellar convection zone

Solar model — Observed frequencies



Canuto-Mazzitelli



Mixing Length Theory

NONRADIAL OSCILLATIONS OF STARS

- The equations for nonradial stellar oscillations with simple boundary conditions were shown to form a Hermitian eigenvalue problem and hence follow a variational formulation:

$$\sigma^2 \int_V \rho |\boldsymbol{\xi}|^2 \, d\mathbf{x} = \int_V \left[\gamma p (\nabla \cdot \boldsymbol{\xi})^2 + \frac{2}{r} \frac{dp}{dr} (\mathbf{x} \cdot \boldsymbol{\xi}) \nabla \cdot \boldsymbol{\xi} \right] \, d\mathbf{x}$$

$$+ \int_V \frac{(\mathbf{x} \cdot \boldsymbol{\xi})^2}{r^2 \rho} \frac{d\rho}{dr} \frac{dp}{dr} \, d\mathbf{x}$$

$$- G \int_V \int_V \frac{(\nabla \cdot \rho \boldsymbol{\xi})_x (\nabla \cdot \rho \boldsymbol{\xi})_{x'}}{|\mathbf{x} - \mathbf{x}'|} \, d\mathbf{x} \, d\mathbf{x}'$$

- These equations were used to obtain the normal modes of oscillations in terms of vector spherical harmonics:

$$\xi_r = \frac{\psi(r)}{r^2} Y_\ell^m(\theta,\phi)$$

$$\xi_\theta = \frac{1}{\ell(\ell+1)r} \frac{d\chi(r)}{dr} \frac{\partial Y_\ell^m(\theta,\phi)}{\partial\theta}$$

$$\xi_\phi = \frac{1}{\ell(\ell+1)r\sin\theta} \frac{d\chi(r)}{dr} \frac{\partial Y_\ell^m(\theta,\phi)}{\partial\phi}$$

- The formulation for radial oscillations ($\chi(r) = 0, \ell = 0$) was also obtained.

- Chandra also obtained the so-called Kelvin mode, or the f-mode using $\psi = \chi = r^{\ell+1}$ to get

$$\sigma^2 = \frac{2\ell(\ell-1)}{2\ell+1} G \frac{\int_0^R \rho r^{2\ell-3} M(r)\, dr}{\int_0^R \rho r^{2\ell}\, dr}$$
$$\approx \ell \frac{GM}{R^3} = gk \qquad (\ell \to \infty)$$

- The frequencies of f-modes have been used to estimate solar radius.

- On formation of close binaries by two-body tidal capture
  W. H. Press, S. A. Teukolsky (1977) ApJ 213, 183

- They used mechanism of Fabian, Pringle and Rees (1975) where two stars in hyperbolic orbit have a close encounter, which leads to a elliptic orbit when sufficient energy is transferred to nonradial oscillations in these stars. This mechanism was proposed for formation of X-ray binaries in globular clusters.

- They formulated the problem as a forced oscillator with tidal forces forming the forcing term:

$$\left(\mathcal{L} - \rho\omega^2\right)\boldsymbol{\xi} = \rho\nabla U$$

- By expanding the tidal forcing in terms of eigenfunctions, they obtained the amplitude of normal modes that are excited.

- Rapidly rotating neutron star models

- J. L. Friedman, J. R. Ipser, L. Parker
  ApJ (1986) 304, 115

- They investigated the structure of rapidly rotating relativistic models for various nuclear matter EOS. They obtained upper limit on rotation rate for different models.

- On the stability of differentially rotating bodies
- D. Lynden-Bell and J. P. Ostriker
  MNRAS (1967) 136, 293

- They generalised the variational principle to a differentially rotating self-gravitating body. Clement (1964, ApJ 140, 1045) had generalised the variational principle to uniformly rotating star. This could be used to study stability of differentially rotating stars.

Application to stellar oscillations

- Variational formulation has been used to study the effect of small perturbations to basic spherically symmetric stellar model. The perturbations could be due to other forces, e.g., rotation or magnetic field or due to perturbation in the stellar models or due to truncation error in numerical calculation of frequencies.

- We can write the perturbed operator as $\mathcal{L} + \delta\mathcal{L}$ giving

$$(\mathcal{L} + \delta\mathcal{L})\boldsymbol{\xi} = \rho(\omega^2 + \delta\omega^2)\boldsymbol{\xi}$$

and in a degenerate perturbation theory we get

$$\delta\omega_\lambda^2 = \frac{\langle \boldsymbol{\xi_\lambda}^* \delta\mathcal{L}\boldsymbol{\xi_\lambda} \rangle}{\langle \boldsymbol{\xi_\lambda}^* \rho \boldsymbol{\xi_\lambda} \rangle}$$

- Since solar oscillation frequencies have been measured to very high accuracy, the solar model frequencies also need to be calculated to even better accuracy for proper comparison.

- The second order finite difference representation that is normally used to solve the eigenvalue problem does not give the required accuracy unless the number of mesh points is larger than 10000.

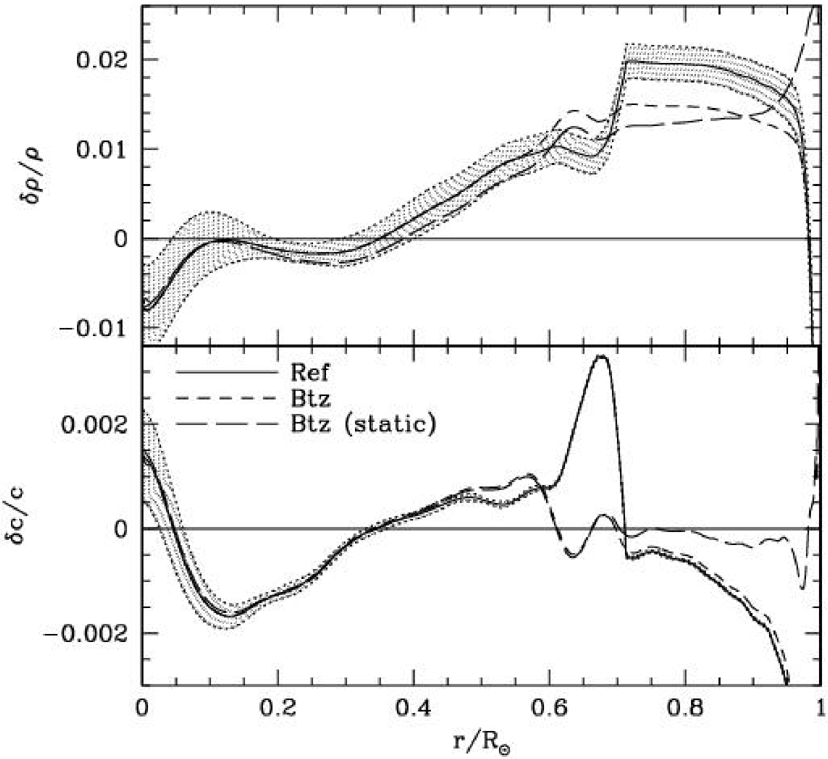
- The truncation error in this difference approximation is treated as perturbation and the correction to the frequency calculated using the variational principle.

- If we consider perturbation in a stellar model, the change can be expressed in terms of perturbation in the sound speed and density, giving

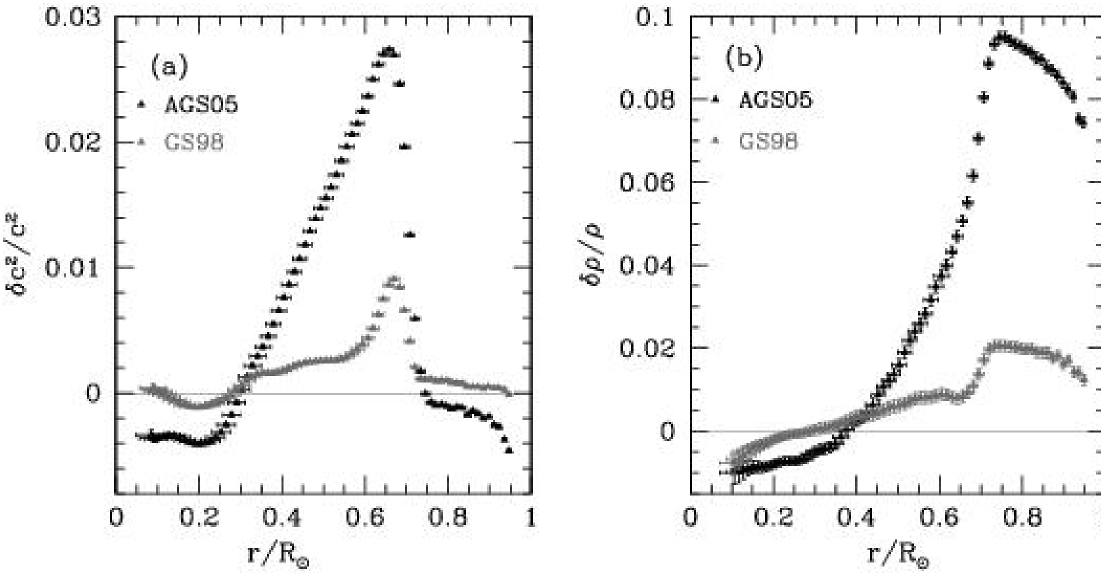
$$\frac{\delta\nu_{n\ell}}{\nu_{n\ell}} = \int_0^R \mathcal{K}^{n\ell}_{c^2,\rho}(r)\frac{\delta c^2}{c^2}(r)dr + \int_0^R \mathcal{K}^{n\ell}_{\rho,c^2}(r)\frac{\delta\rho}{\rho}(r)dr$$

- It can be shown that c, ρ along with hydrostatic equilibrium are enough to determine the solar model as far as frequencies are concerned. Pressure p and adiabatic index Γ_1 can be determined from c, ρ.

- This equation is used for inversion to calculate c, ρ in the Sun.



Sun – Model (Brun et al. 2002)



Basu & Antia (2008)

- Ritzwoller & Lavely (1991, ApJ 369, 557) treated rotation as a perturbation on spherically symmetric stellar model to calculate the first order effect of rotation, arising from the Coriolis force

$$\delta\mathcal{L} = -2i\omega\rho\mathbf{v}_{\mathrm{rot}} \cdot \nabla\boldsymbol{\xi}$$

  to calculate the frequency splitting due to rotation.

- The frequency splitting can be expressed as

$$\nu_{n,\ell,m} = \nu_{n,\ell} + \sum_{j=1}^{M} a_j^{(n,\ell)} P_j(m)$$

  and the rotation velocity is decomposed as

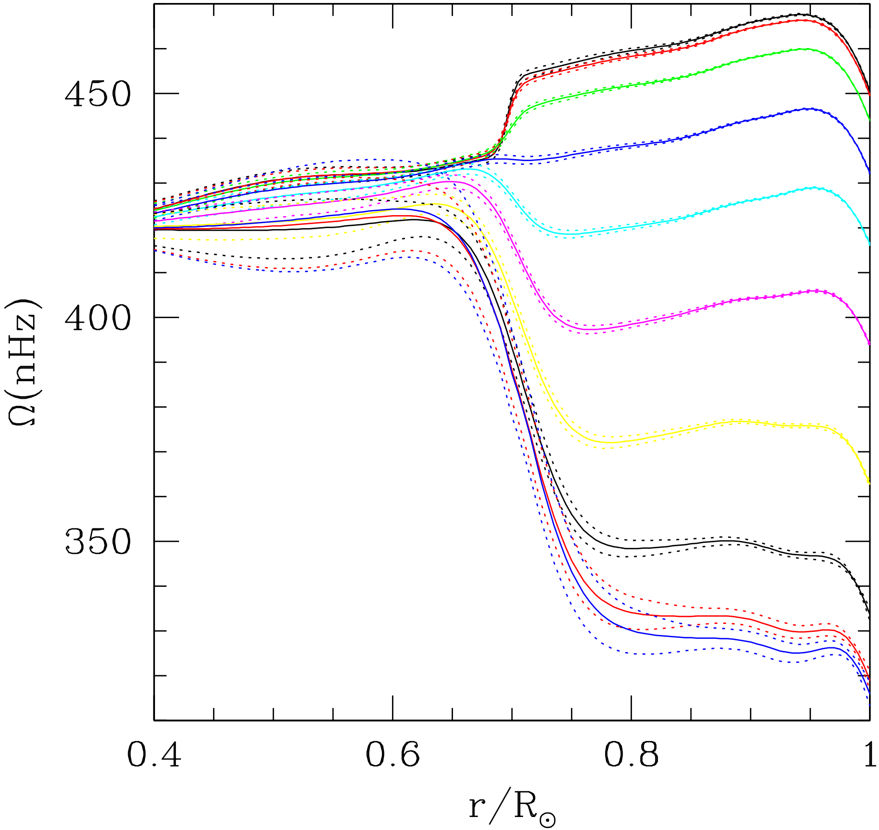
$$v_\phi(r,\theta) = -\sum_{j=1}^{M} w_{2j-1}(r) \frac{\partial Y_{2j-1}^0}{\partial\phi}$$

- With these decomposition, using the variational principle the splitting coefficients are given by

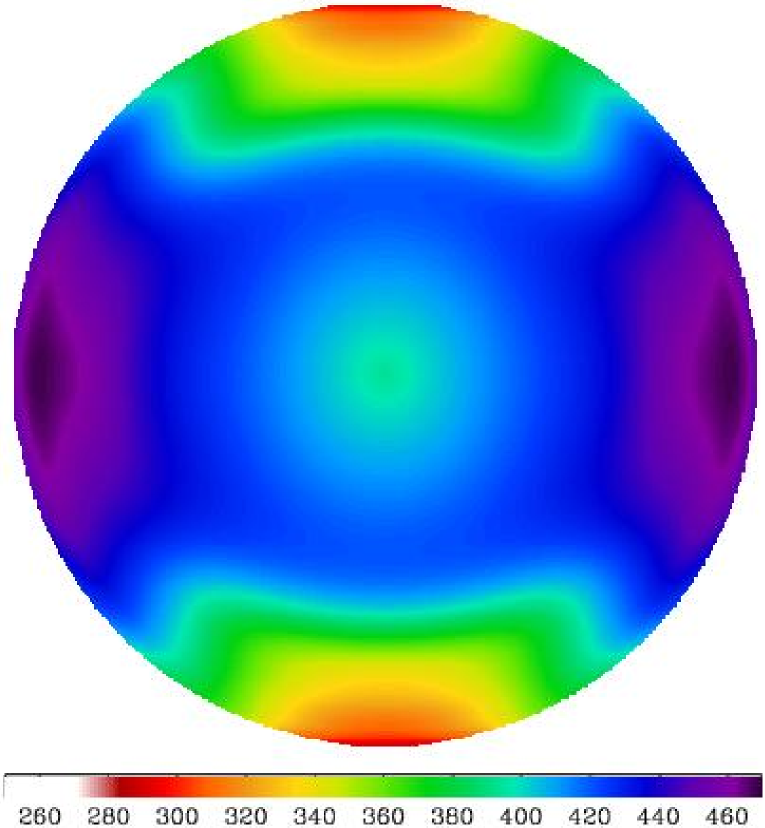
$$
\begin{aligned}
a_j^{(n,\ell)} &= \int_0^R w_j(r)\mathcal{K}_j^{(n,\ell)}(r)r^2\,dr \\
&= \int_0^R \int_{-1}^1 \Omega(r,\theta)\mathcal{K}_j^{(n,\ell)}(r,\theta)\,dr\,d\cos\theta
\end{aligned}
$$

  This equation can be used for inversion of rotation rate from observed splitting coefficients.

- The splitting coefficients are sensitive only to the North–South symmetric component of rotation rate and hence that is the only component that can be determined.



450
400
350
Ω(nHz)
0.4
0.6
0.8
1
r/R⊙



260
280
300
320
340
360
380
400
420
440
460

- Shear layer at the surface
- Tachocline
- Global quantities: (Pijpers 1998)

$$\begin{aligned} \text{Angular Momentum: } H &= (190.0 \pm 1.5) \times 10^{46} \text{ gm cm}^2 \text{ s}^{-1} \\ \text{Kinetic Energy: } T &= (253.4 \pm 7.2) \times 10^{40} \text{ gm cm}^2 \text{ s}^{-2} \\ \text{Quadrupole Moment: } J_2 &= (2.18 \pm 0.06) \times 10^{-7} \, . \end{aligned}$$

J_2 will cause precession of perihelion of Mercury by $0.03''$ per century.

- Gough & Thompson (1990, MNRAS 242, 25) extended the study to include second order effect of rotation and magnetic field

$$(\mathcal{L} - \rho\omega^2)\boldsymbol{\xi} = \omega\mathcal{M}\boldsymbol{\xi} + \mathcal{N}\boldsymbol{\xi} + \mathcal{B}\boldsymbol{\xi}$$

- The second order effects contribute to the even order splitting coefficients and can be separated from rotation. They also calculated the effect of departure from spherical symmetry due to rotation.

- The magnetic field was considered to be axisymmetric, either toroidal

$$\mathbf{B} = \left[0, 0, a(r)\frac{\mathrm{d}}{\mathrm{d}\theta}P_k(\cos\theta)\right]$$

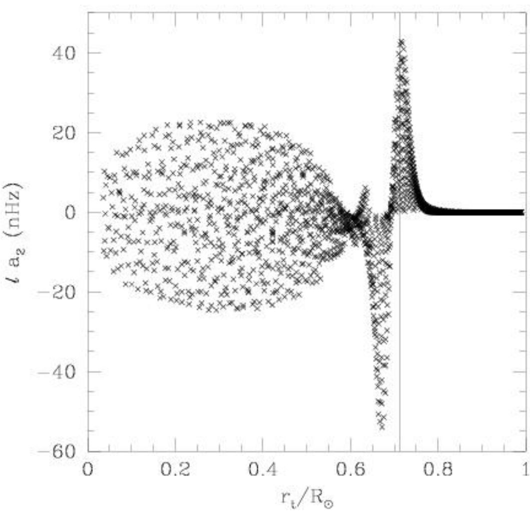
or poloidal

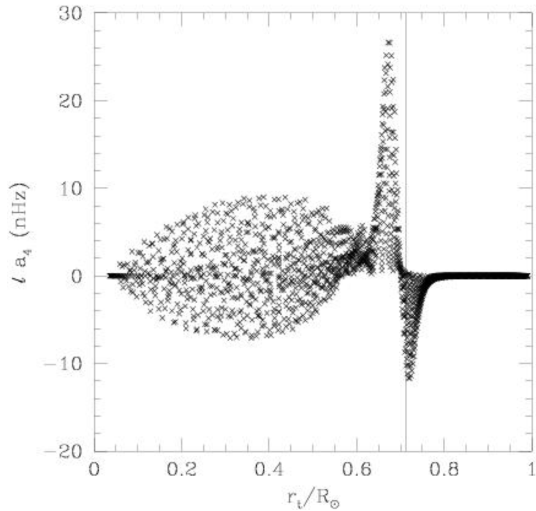
$$\mathbf{B} = \left[k(k+1)\frac{b(r)}{r^2}P_k(\cos\theta), \frac{1}{r}\frac{\mathrm{d}b}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}\theta}P_k(\cos\theta), 0\right]$$

- Using a toroidal field with

$$a(r) = \begin{cases} \sqrt{8\pi p_0\beta_0}(1-(\frac{r-r_0}{d})^2) & \text{if } |r-r_0| \leq d \\ 0 & \text{otherwise} \end{cases}$$

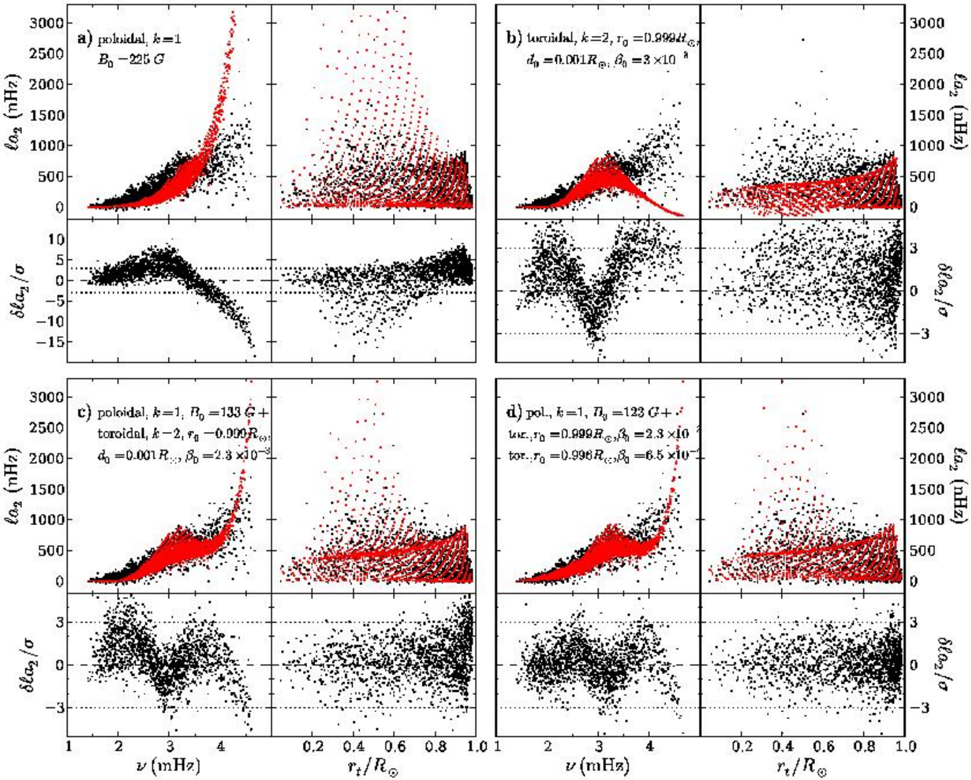
- For $k = 2$, $\beta_0 = 10^{-4}$, $r_0 = 0.713R_\odot$, $d = 0.02R_\odot$, the splitting coefficients are



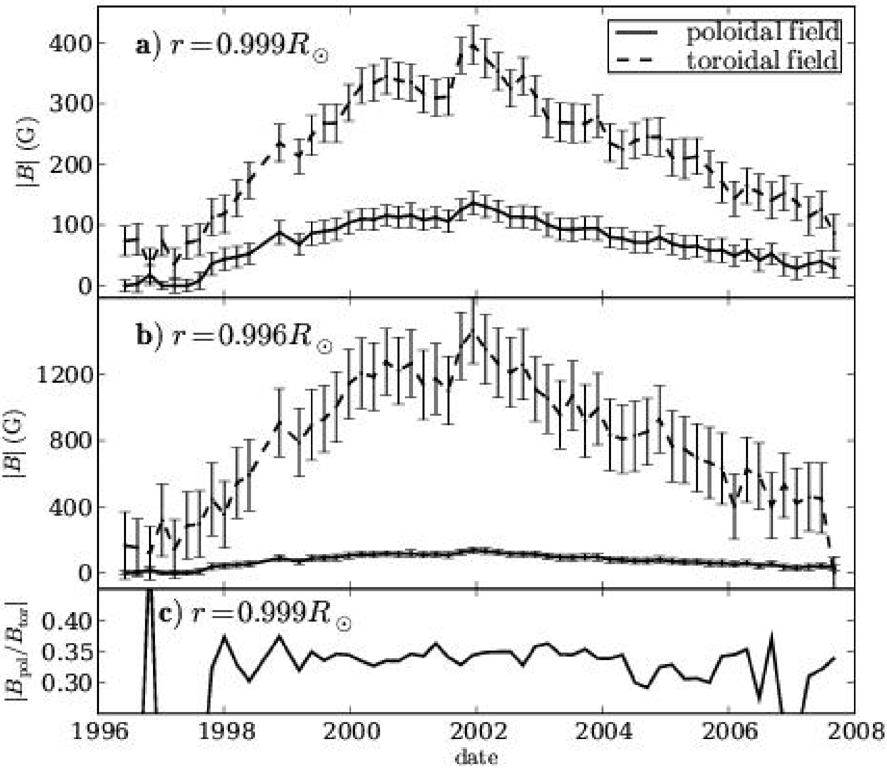


Turning point r_t is given by $\dfrac{\ell(\ell+1)c^2(r_t)}{r_t^2} = \omega^2$

Antia et al. (2000)



Baldner et al. (2009)



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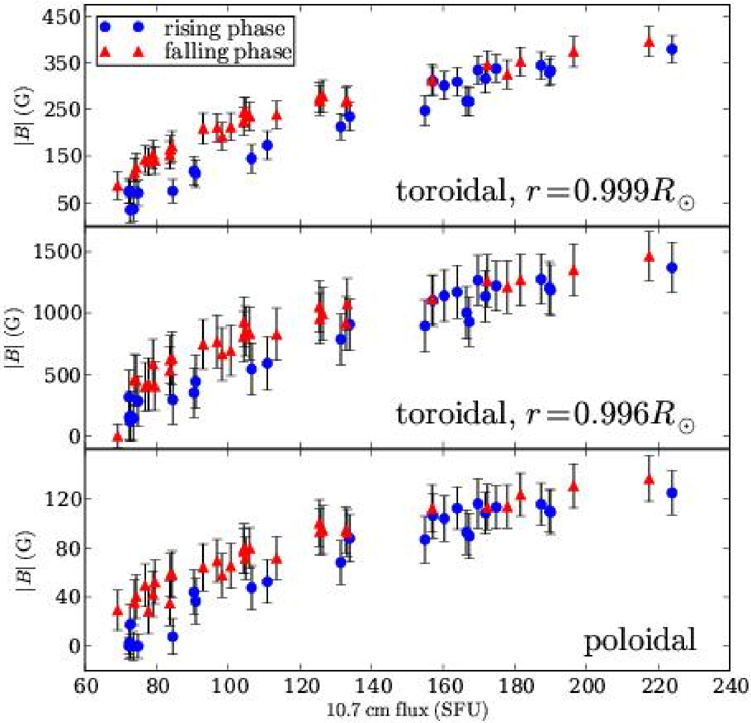


Baldner et al. (2009)

- Contribution to frequency shift from meridional flow, or the North–South antisymmetric component of rotaton vanishes in this limit and one has to use the quasi-degenerate perturbation theory (Lavely & Ritzwoller 1992).
- In this case the eigenfunction is written as the sum of eigenfunctions of spherically symmetric model with close frequencies

$$\boldsymbol{\xi}'_{\boldsymbol{k}} = \sum_{k'} a_{k'} \boldsymbol{\xi}_{\boldsymbol{k'}}$$

and the equations are given by

$$(\mathcal{L} - \rho {\omega'_k}^2)\boldsymbol{\xi}'_{\boldsymbol{k}} + \delta\mathcal{L}\boldsymbol{\xi}'_{\boldsymbol{k}} = 0$$

which gives a set of equations:

$$\sum_{k'} a_{k'} \left[H_{jk'} + \delta_{jk'}(\omega^2_{k'} - \omega^2_k)\right] = a_j({\omega'_k}^2 - \omega^2_k)$$

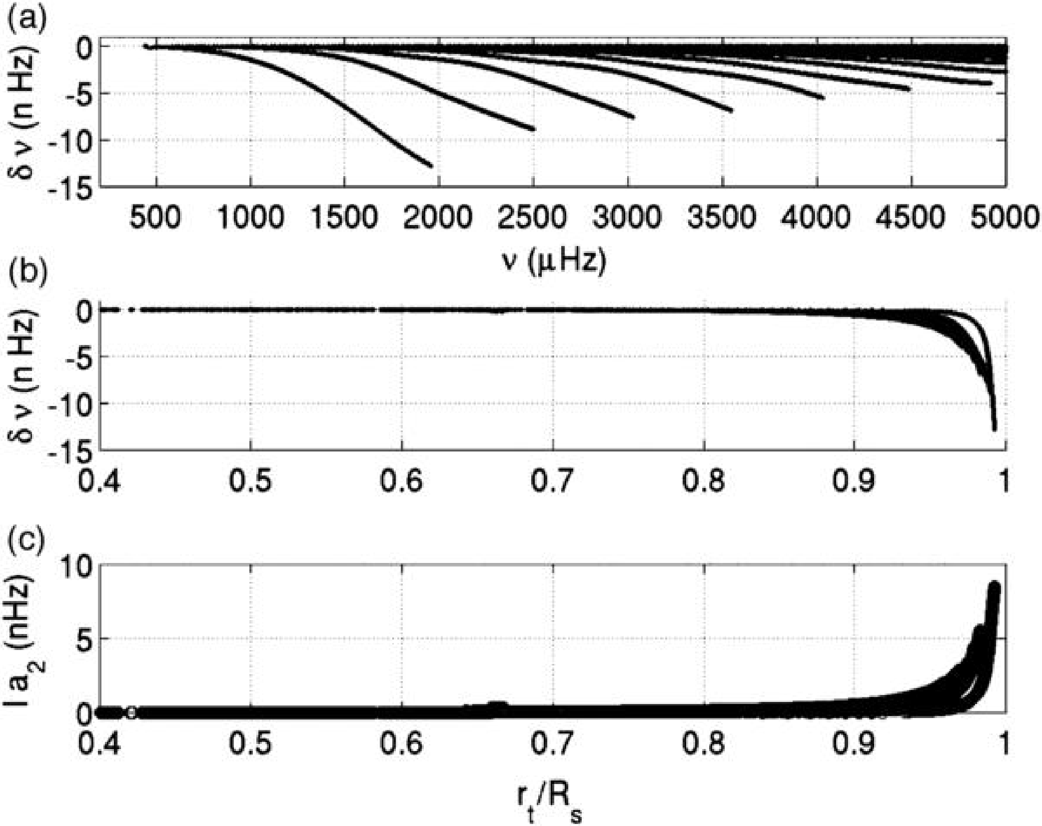
where $H_{jk} = \langle \boldsymbol{\xi_j}^* \delta\mathcal{L}\boldsymbol{\xi_k} \rangle$.

- The axisymmetric meridional velocity is of the form

$$\mathbf{v}_s(r,\theta) = \left[u_s(r)P_s(\cos\theta), v_s(r)\frac{\mathrm{d}}{\mathrm{d}\theta}P_s(\cos\theta), 0\right]$$

$$v_s(r) = \frac{1}{\rho r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\rho r^2 u_s(r)}{s(s+1)}\right)$$

$$u_s(r) = \begin{cases} u_0\frac{4(R-r)(r-r_b)}{(R-r_b)^2} & \text{if } r_b \le r \le R \\ 0 & \text{otherwise} \end{cases}$$



Chatterjee & Antia (2009)