Statistical Mechanics of Gravitating Systems

...and some curious history of Chandra's rare misses!

T. Padmanabhan

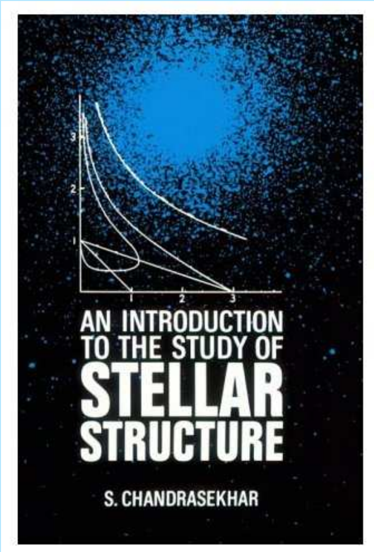
(IUCAA, Pune, INDIA)

Chandra Centenary Conference

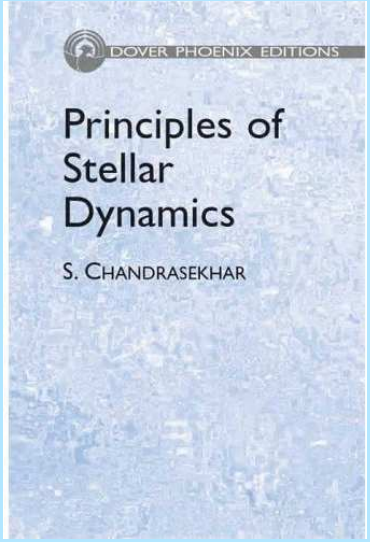
Bangalore, India

8 December 2010

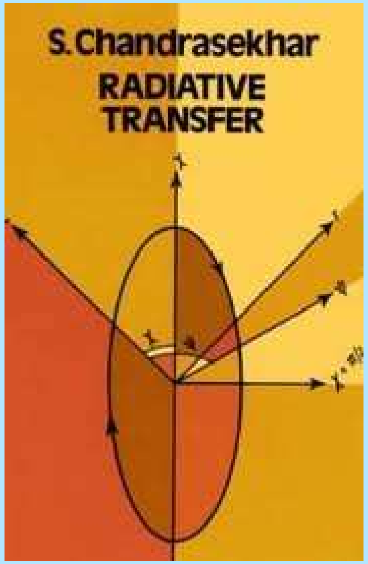
Phases of Chandra's life



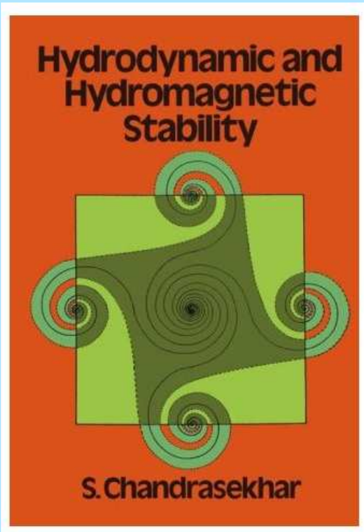
[1939]



[1942]



[1950]



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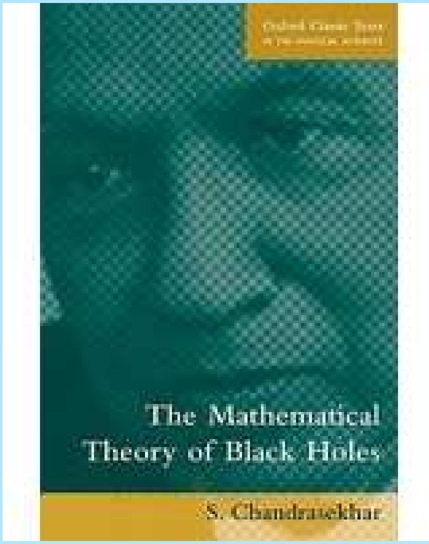
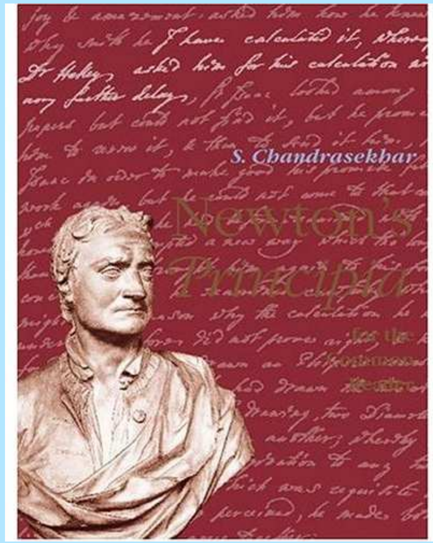


[1983]



[1995]

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UNIVERSAL POWER SPECTRUM,
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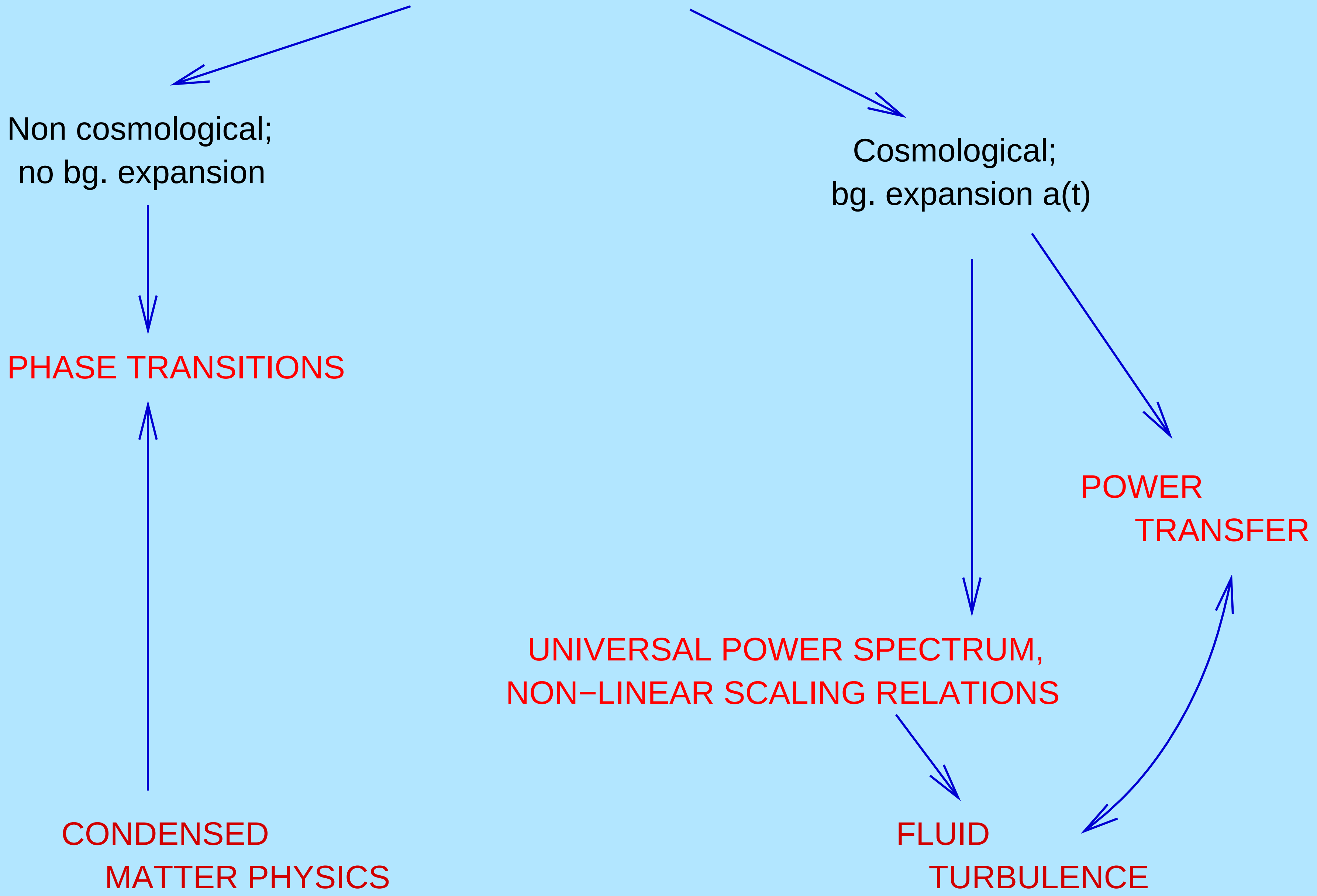

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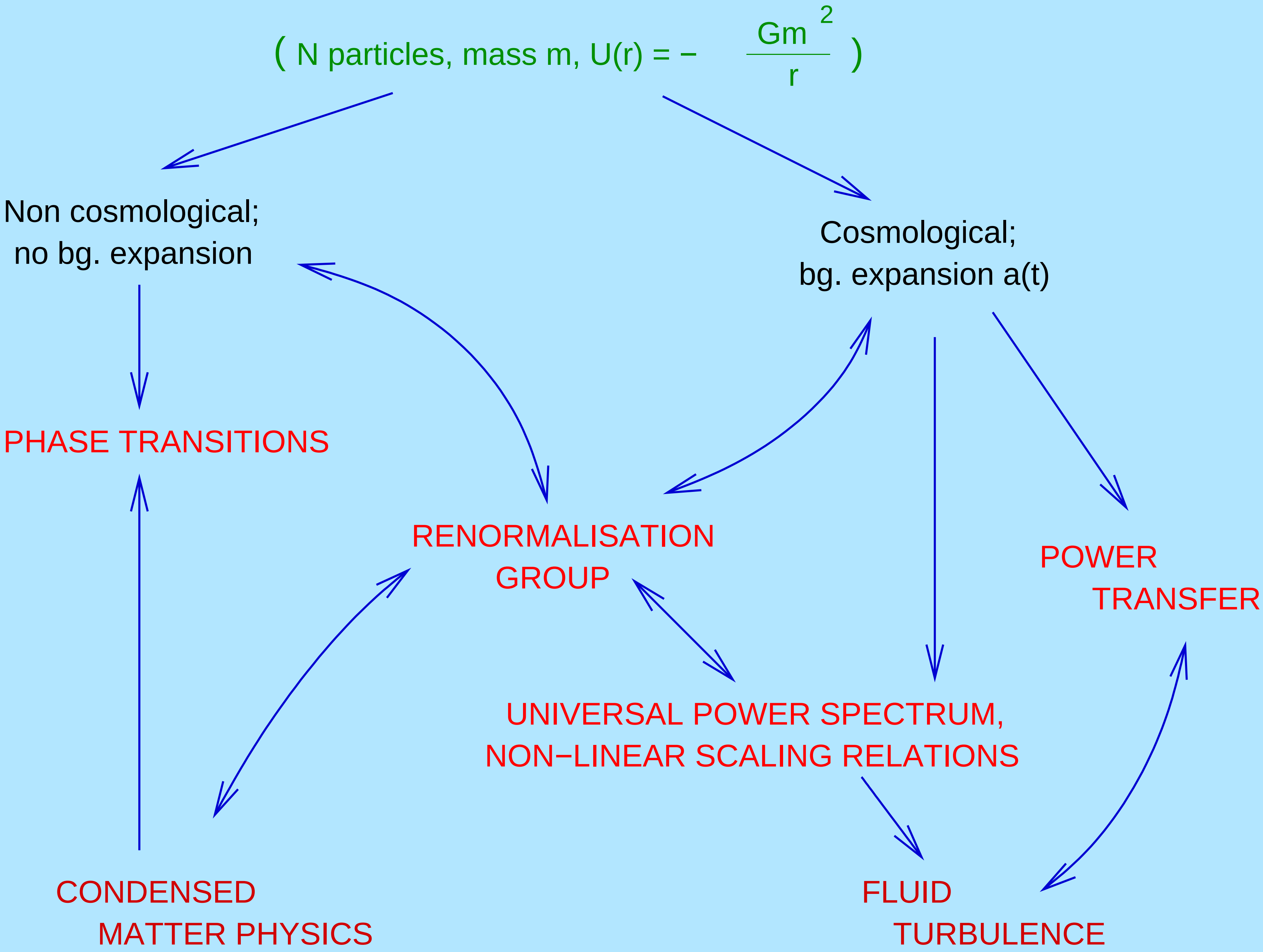

PLAN OF THE TALK

- FINITE SELF-GRAVITATING SYSTEMS OF PARTICLES
  - *General features*
  - *Mean field description: Isothermal sphere*
  - *Antonov instability*
- RELAXATION TIME AND DYNAMICAL FRICTION
  - *Chandra's contribution*
  - *Historical background*
- GRAVITATING PARTICLES IN EXPANDING BACKGROUND
  - *General features, Open questions*
  - *Power transfer, Inverse cascade*
  - *Universality*

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- If energy is extensive, these descriptions are equivalent for most systems to $\mathcal{O}(\ln N/N)$.

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- The $g(E)$ and $S(E)$ requires short-distance cutoff for finiteness.

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- Increasing R increases the range over which the specific heat is negative.

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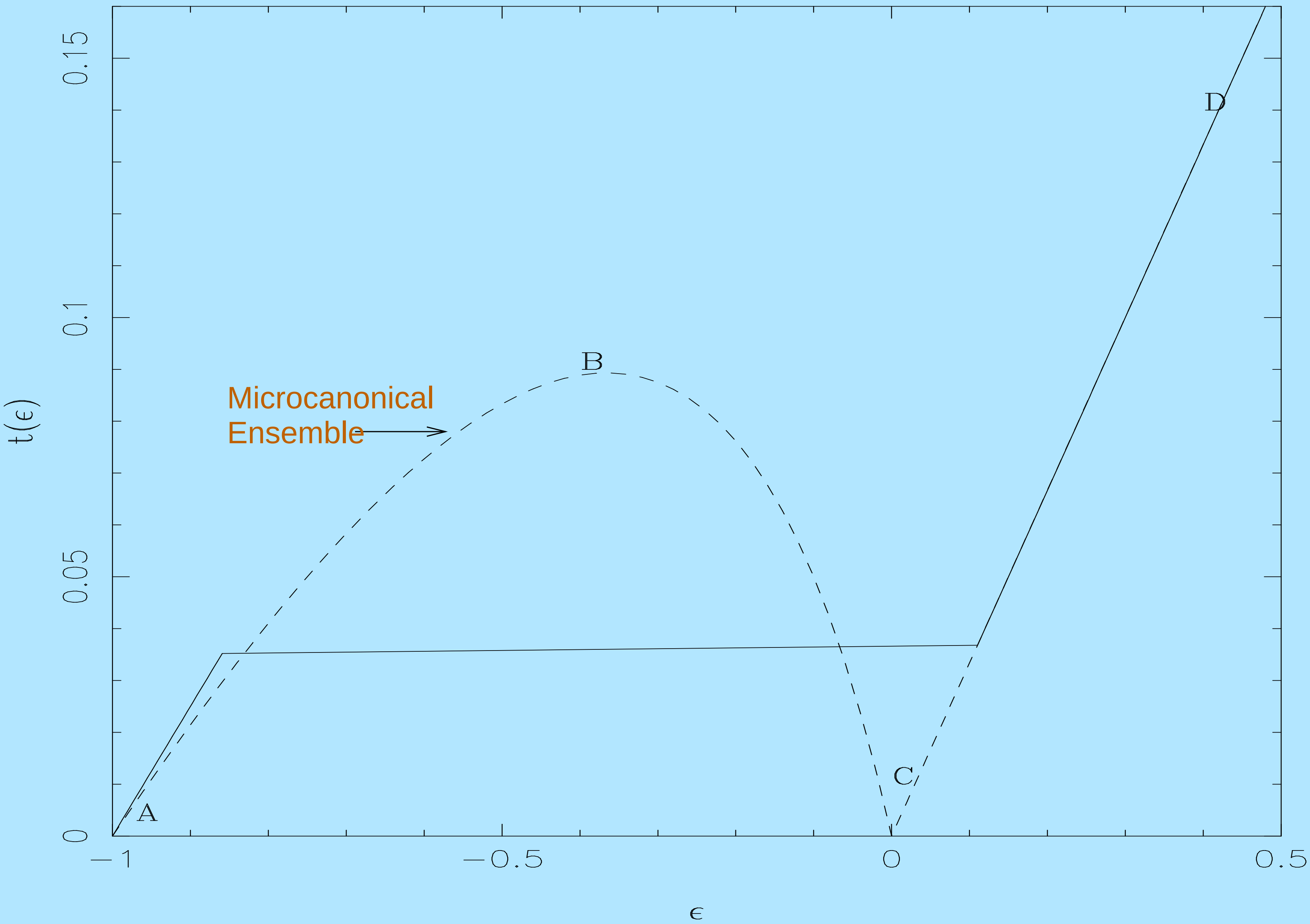
T. Padmanabhan, Physics Reports, **188**, 285 (1990).



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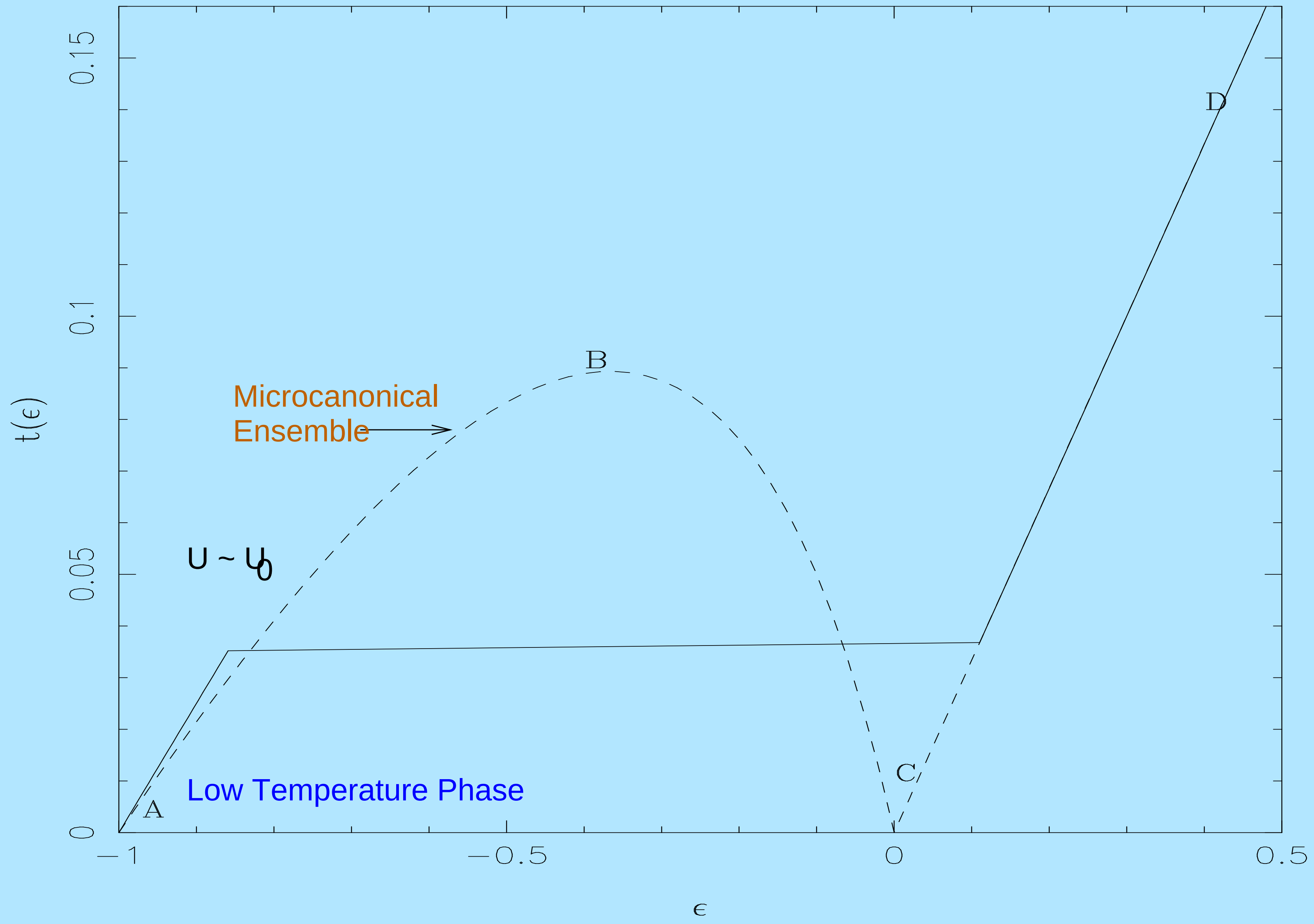

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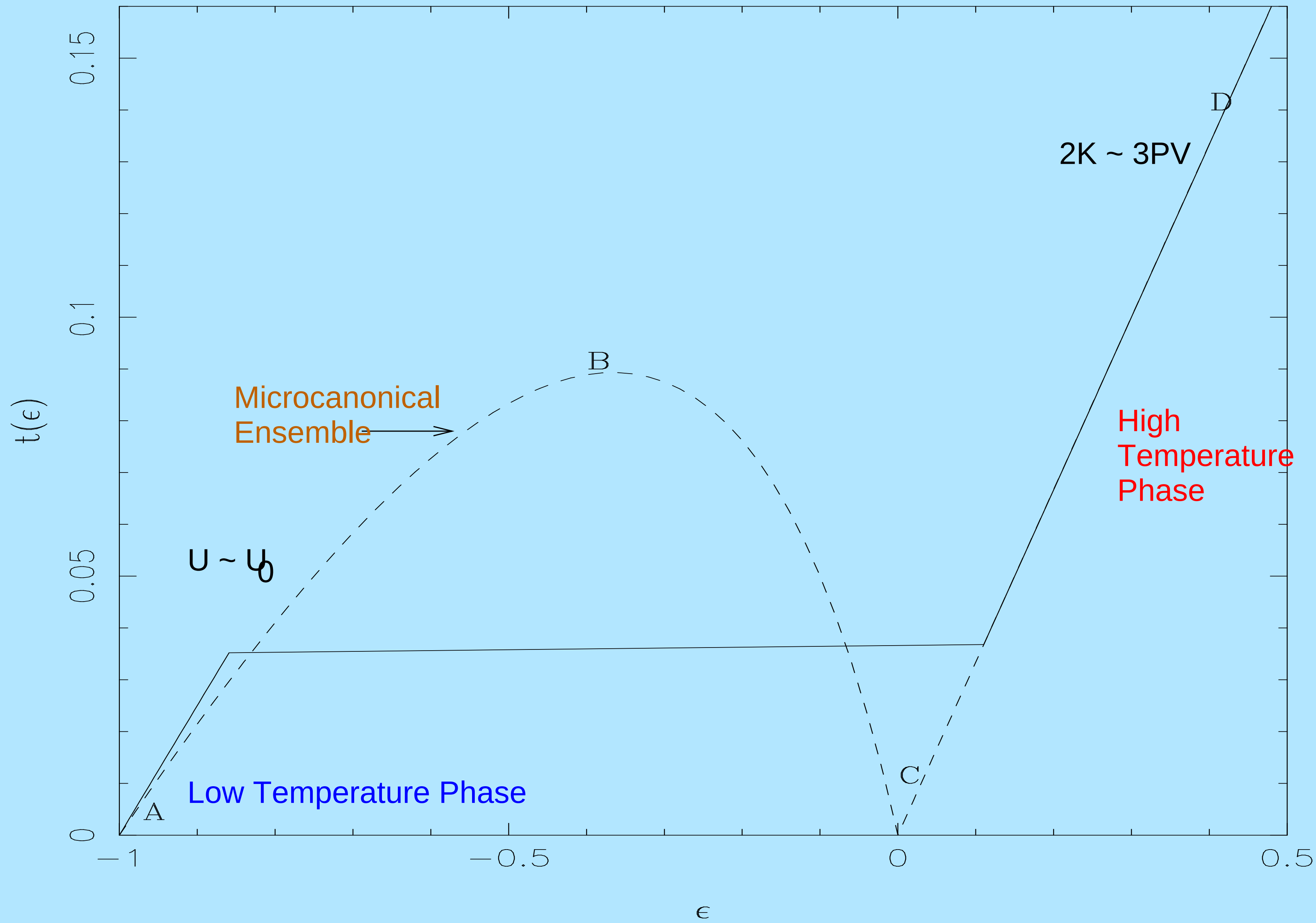

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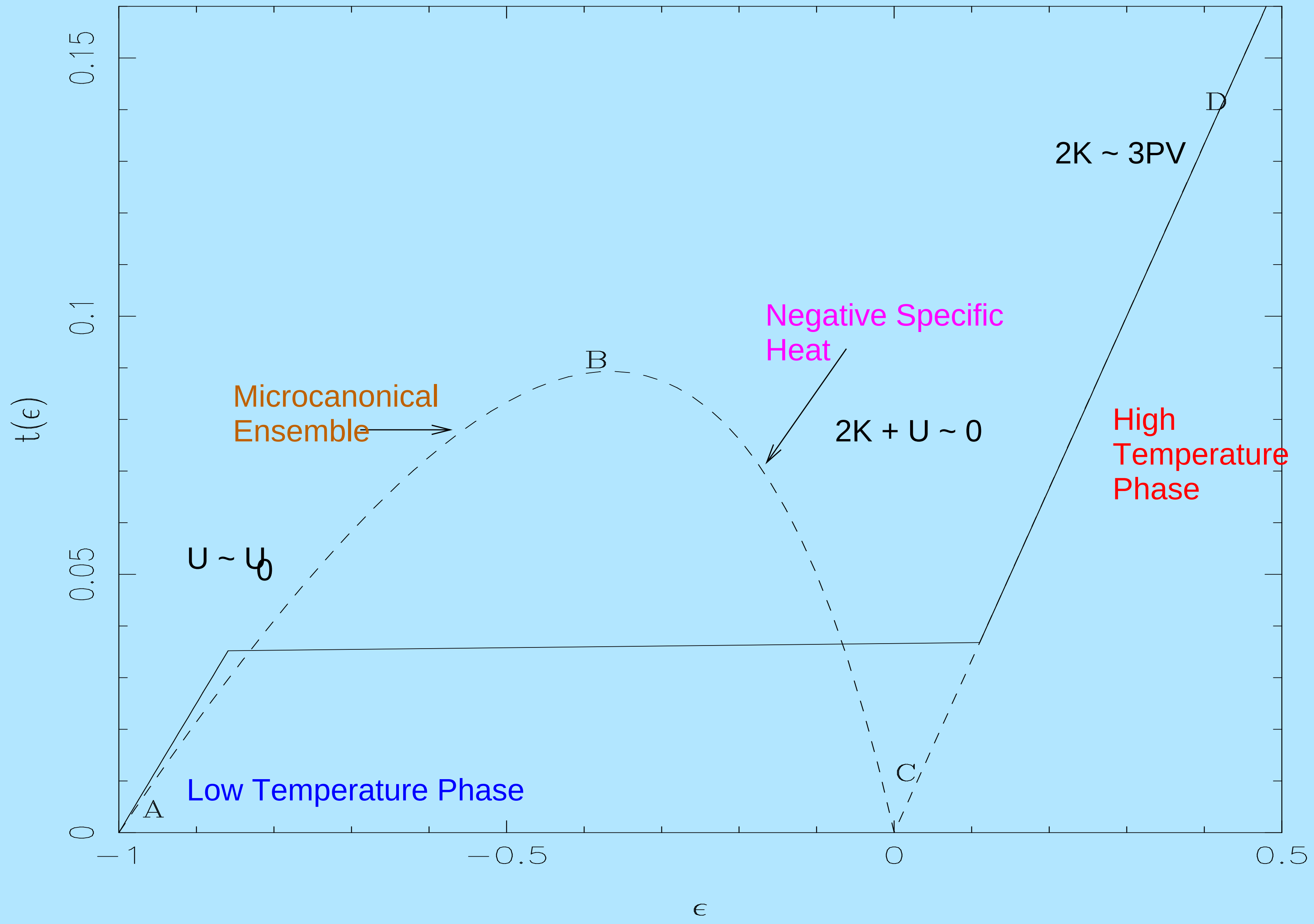

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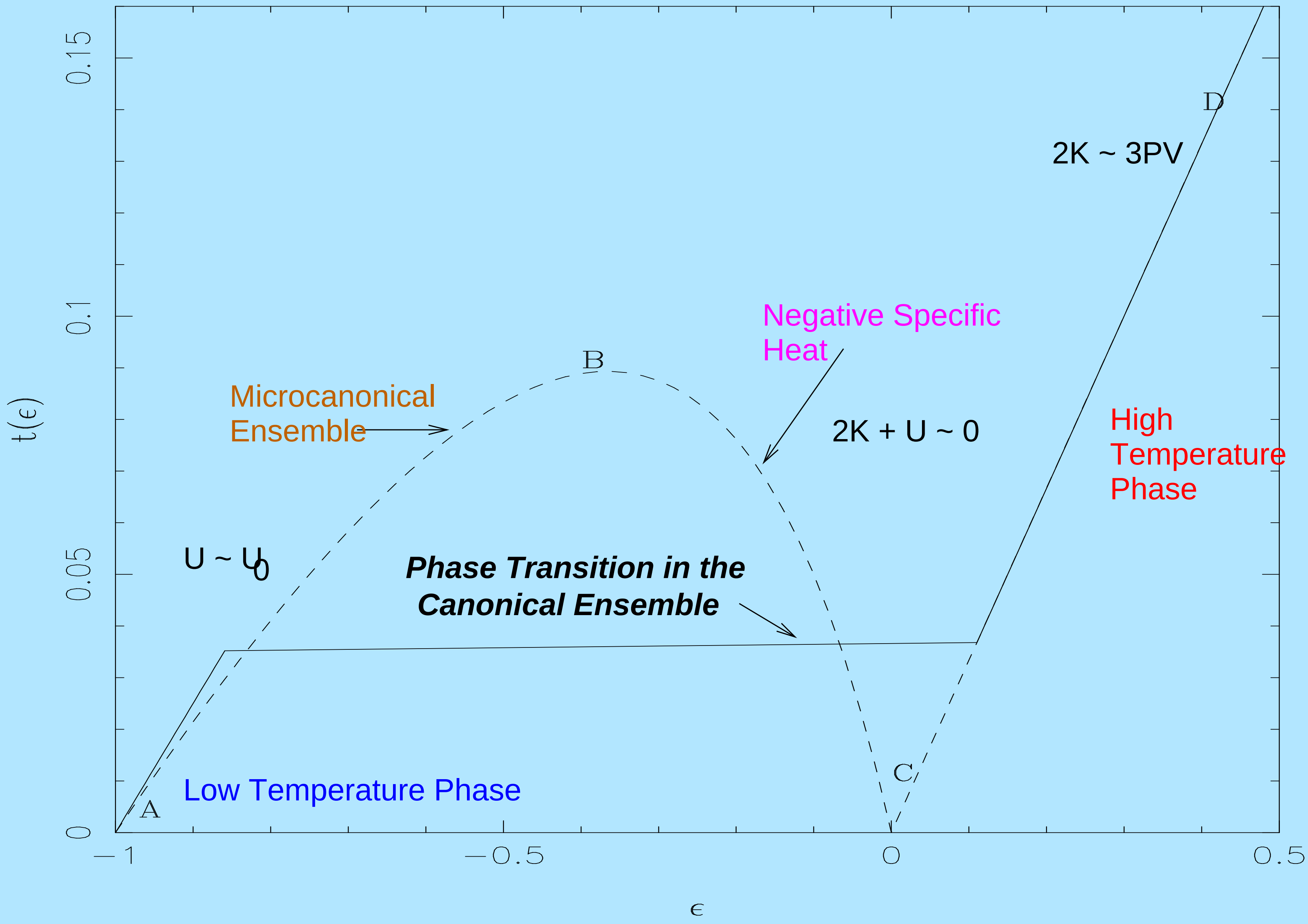

Mean field description of many-body systems

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- A system of N particles interacting through the two-body potential $U(\mathbf{x},\mathbf{y})$. The entropy S of this system

$$\mathrm{e}^S = g(E) = \frac{1}{N!}\int d^{3N}x d^{3N}p\,\delta(E-H) \propto \frac{1}{N!}\int d^{3N}x \left[E - \frac{1}{2}\sum_{i\neq j} U(\mathbf{x}_i,\mathbf{x}_j)\right]^{\frac{3N}{2}}$$

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- Divide spatial volume V be divided into M cells of equal size. Integration over the particle coordinates $(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)$= sum over the number of particles n_a in the cell centered at $\mathbf{x}_a$ (where $a = 1, 2, ..., M$).

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- This gives:

$$\mathrm{e}^S \approx \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \dots \sum_{n_M=1}^{\infty} \delta\left(N - \sum_a n_a\right) \exp S[\{n_a\}]$$

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- The mean field approximation: retain in the sum only the term for which the summand reaches the maximum value

$$\sum_{\{n_a\}} \mathrm{e}^{S[n_a]} \approx \mathrm{e}^{S[n_{a,\max}]}$$

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- For gravitational interactions *without* a short distance cut-off, the quantity e^S is divergent. A short distance cut-off is needed to justify the entire procedure.

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  V.A. Antonov, V.A. : Vest. Leningrad Univ. **7**, 135 (1962); Translation: IAU Symposium **113**, 525 (1985).

  T.Padmanabhan: Astrophys. Jour. Supp. , **71**, 651 (1989).

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with dimensionless variables:

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- The singular point is at: $u_s = 1$, $v_s = 2$; corresponding to the solution $n = (2/x^2), m = 2x$.

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$$v \equiv \frac{m}{x}; \qquad u \equiv \frac{nx^3}{m} = \frac{nx^2}{v}.$$

$$\frac{u}{v}\frac{dv}{du} = -\frac{(u-1)}{(u+v-3)}; \qquad v(u=3) = 0; \qquad v'(u=3) = -\frac{5}{3}$$

- The singular point is at: $u_s = 1$, $v_s = 2$; corresponding to the solution $n = (2/x^2), m = 2x$.

- All solutions tend to this asymptotically for large r by spiralling around the singular point in the $u - v$ plane.

LANE-EMDEN VARIABLES

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- Finite total mass for the system requires a large distance cut-off at some $r = R$.

singular solution, oscillating with respect to it and intersecting it at points which asymptotically increase geometrically in the ratio $e^{2\pi/\sqrt{7}}$.

27. *Discussion of the isothermal equation in the* (u, v) *plane.*—We shall conclude our discussion of the isothermal equation by a brief description of the solution-curves in the (u, v) plane.

Our variables are

$$u = \frac{\xi e^{-\psi}}{\psi'} ; \qquad v = \xi\psi' , \tag{452}$$

where u and v satisfy the first-order equation

$$\frac{u}{v}\frac{dv}{du} = -\frac{u-1}{u+v-3} . \tag{453}$$

a) The locus of points at which the curves have horizontal tangents is given by

$$u = 1 , \tag{454}$$

which is a line parallel to the v-axis.

b) The locus of points at which the curves have vertical tangents is given by

$$u + v = 3 . \tag{455}$$

c) The two loci (454) and (455) intersect at the point

$$u_s = 1 ; \qquad v_s = 2 . \tag{456}$$

It is clear, therefore, that the E-curve starts at the point ($u = 3$, $v = 0$) with a negative slope of $5/3$ and approaches the point ($u = 1$, $v = 2$) by spiraling around it (cf. Fig. 20).

e) All the other solutions also spiral around this point, and it is clear that along these curves $v \to 0$ as $u \to \infty$. This arises because, as we have already seen, these solutions correspond to a ρ which vanishes at $\xi = 0$ and at $\xi = \infty$, and hence ψ' must vanish for some finite ξ; for this value of ξ, $v = 0$ and $u = \infty$.

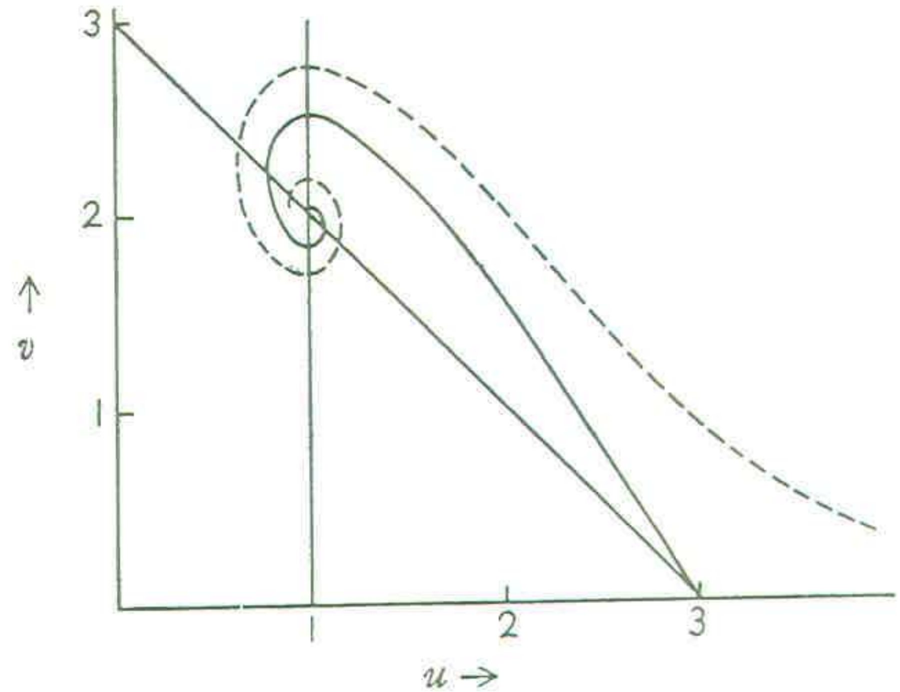


FIG. 20.—The (u, v) curves $(n = \infty)$

GASKUGELN

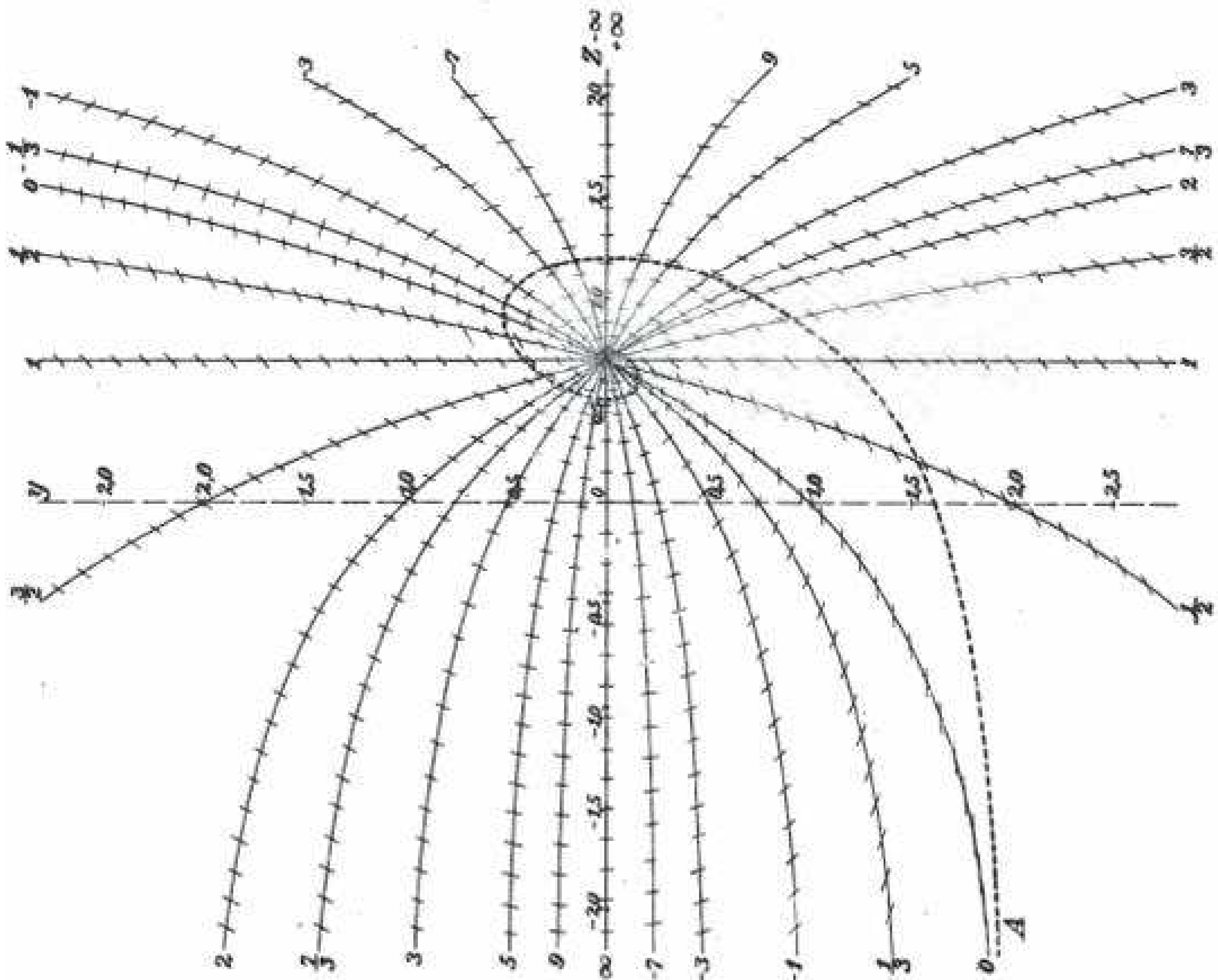
ANWENDUNGEN DER
ECHANISCHEN WÄRMETHEORIE
F KOSMOLOGISCHE UND METEOROLOGISCHE PROBLEME

VON

DR. R. EMDEN
PRIVATDOZENT FÜR PHYSIK UND METEOROLOGIE
AN DER KGL. TECHNISCHEN HOCHSCHULE IN MÜNCHEN

MIT 24 FIGUREN, 12 DIAGRAMMEN UND 5 TAFELN IM TEXT

LEIPZIG UND BERLIN
DRUCK UND VERLAG VON B. G. TEUBNER
1907



im Kapitel IX, § 9 b) nachwiesen, $-\mathfrak{r}_1 \frac{dv_1}{d\mathfrak{r}_1}$ mit wachsendem $\mathfrak{r}_1$ um den Wert 2. Wir deuten in der Fig. 8 den Gang von $-\mathfrak{r}_1 v_1'$ als

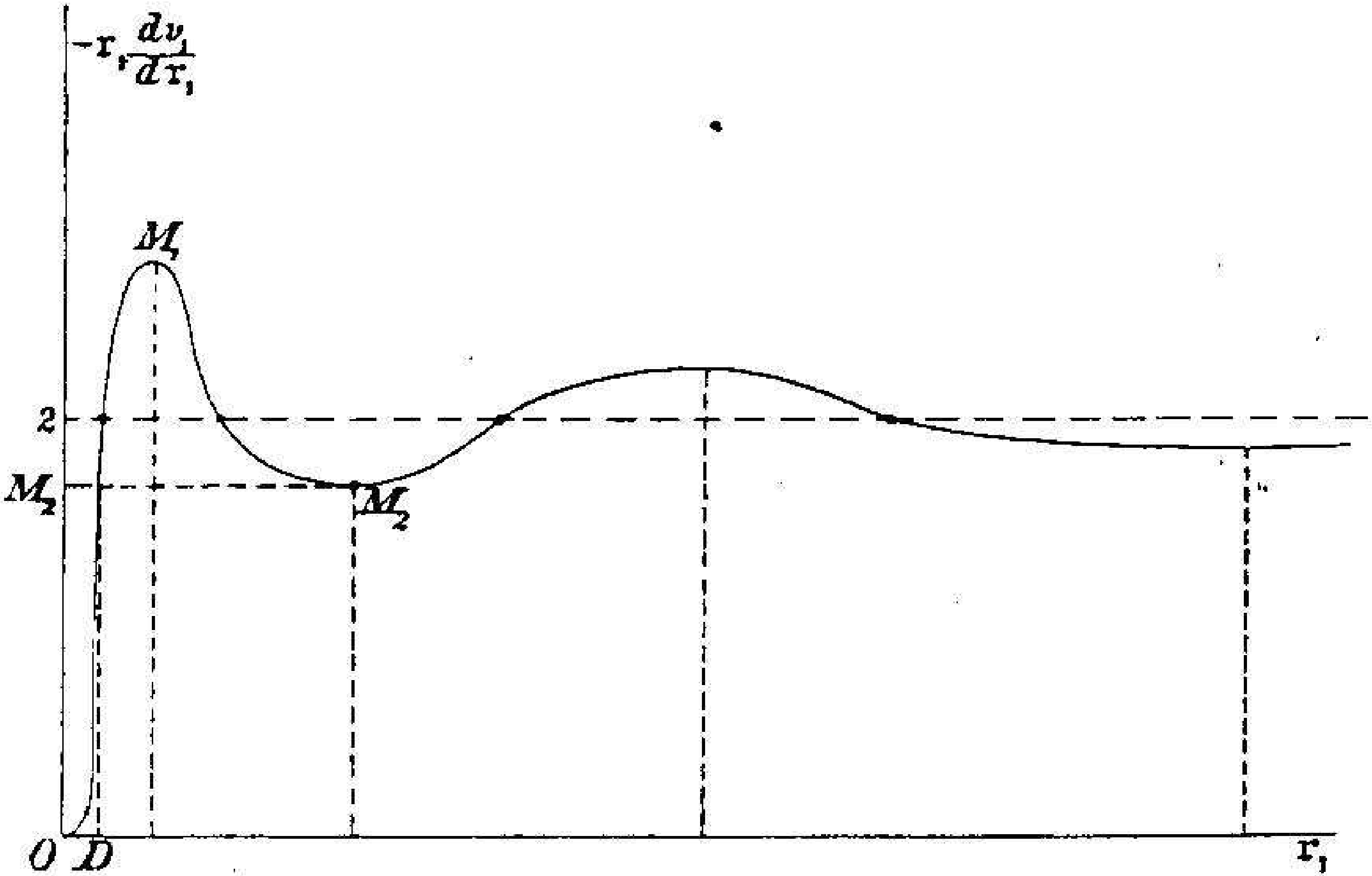


Fig. 8.

Funktion von $\mathfrak{r}_1$ an. In Wirklichkeit folgen die Abszissen der Ordinaten 2 immer genauer einer geometrischen Progression mit dem

[D.Lynden-Bell, R. Wood, (1968), *MNRAS*, **138**, p.495.]

TABLE I

z	$\lvert v_1 \rvert$	ρ_0/ρ_e	Remarks
4·07	1·61	5·0	Turning point of $dM(z)/dz$ (the incremental increase of mass with radius).
4·74	1·93	6·8	Zero of energy for an isolated system of given volume (i.e. the configuration in which the gravitational binding energy just balances the thermal energy).
6·45	2·64	14·1	Minimum of Gibbs free energy for equilibria of systems in contact with a heat bath at constant temperature. Onset of thermal instability at constant pressure (Ebert). Onset of negative* specific heat at constant pressure, C_p. Maximum of isotherm.
7·25	2·93	18·7	Zero of enthalpy for an isolated system at given pressure.
8·99	3·47	32·2	Minimum of Helmholtz free energy for equilibria of systems in contact with a heat bath at constant temperature. Onset of thermal instability at constant volume. Onset of negative* specific heat at constant volume, C_v. Vertical tangent to isotherm. Schönberg–Chandrasekhar limit (approx.).
22·5	5·65	287	Minimum temperature for a given energy. Maximum energy for a given temperature.
25·8	5·96	389	Maximum entropy for an isolated equilibrium configuration at given pressure. Least enthalpy for an equilibrium state of given pressure. Onset of dynamical instability in thermally isolated systems at given pressure. Minimum of adiabat.
34·2	6·55	709	Maximum entropy for an isolated equilibrium configuration of given volume (Antonov). Least total energy (greatest binding energy) for an equilibrium state at given volume. Maximum volume for an equilibrium state of given energy. Onset of the gravo-thermal instability in completely isolated systems. Vertical tangent to adiabat.

Chandra's comments on Emden's work

V. R. Emden. The publication of Emden's *Gaskugeln* marks the end of the first epoch in the study of stellar configurations. Emden's book not only systematizes the earlier work but also contains a fair proportion of new results and a wealth of material, including accurate and extensive tables of the necessary functions. This is not the place to describe the contents of Emden's book, but we may refer specifically to such parts of the analysis contained in our chapter iv which are due to Emden. They are:

1. The use of the (y, z) variables introduced in § 3. Indeed, Emden was the first to reduce the equation to one of the first order.

2. The discovery of the explicit formula for θ_5, independently of Schuster.

3. The discussion given in §§ 9, 10, 11, and 12, and also the discussion in § 13, leading up to the two lemmas which in the form given are due to E. Hopf, *M.N.*, **91**, 653, 1931. These lemmas without rigorous proofs are already implicit in Emden's book (chap. xiii), and Emden himself uses them.

4. The analysis in § 14, and in particular the discovery of the behavior $\theta \sim C/\xi$ for $n < 3$ as $\xi \to 0$.

5. Emden was fully aware of the fact that the E-solutions form a "grid," though the explicit theorem is due to R. H. Fowler.

6. The analysis in § 18. In particular, Emden was the first to isolate the critical role which $n = 3.18767$ (Eq. [255]) plays in the subsequent discussion.

7. The discovery of the behavior near the origin of the general solutions for $3 < n < 5$. In particular, equations (272) and (285), which describe the behavior of θ as $\xi \to 0$.

8. The analysis of § 19.

9. The behavior of the general solutions as $\xi \to \infty$ for $5 < n < \infty$.

10. The use of the (y, z) variables in § 24 for the isothermal gas sphere and the behavior of the general solution as $\xi \to \infty$.

It is thus seen that Emden's own investigations in this field have consisted almost entirely in the discussion of the general solutions; this aspect of his investigations has never been adequately recognized. Though there are a great number of references to *Gaskugeln* in the literature, it is unfortunate that what are generally associated with Emden's name have been derived by the earlier investigators. This is stated, not with a view to minimizing the value of Emden's very great work, but only to draw attention to the fundamental character of his own original contributions.

ENERGY OF THE ISOTHERMAL SPHERE

- The potential and kinetic energies are

$$
\begin{aligned}
U &= -\int_0^R \frac{GM(r)}{r}\frac{dM}{dr}dr = -\frac{GM_0^2}{L_0}\int_0^{x_0} mnx dx \\
K &= \frac{3}{2}\frac{M}{\beta} = \frac{3}{2}\frac{GM_0^2}{L_0}m(x_0) = \frac{GM_0^2}{L_0}\frac{3}{2}\int_0^{x_0} nx^2 dx; \quad x_0 = R/L_0
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$$

- So:

$$
\begin{aligned}
E &= K + U = \frac{GM_0^2}{2L_0}\int_0^{x_0} dx(3nx^2 - 2mnx) \\
&= \frac{GM_0^2}{2L_0}\int_0^{x_0} dx\frac{d}{dx}\{2nx^3 - 3m\} = \frac{GM_0^2}{L_0}\{n_0x_0^3 - \frac{3}{2}m_0\}
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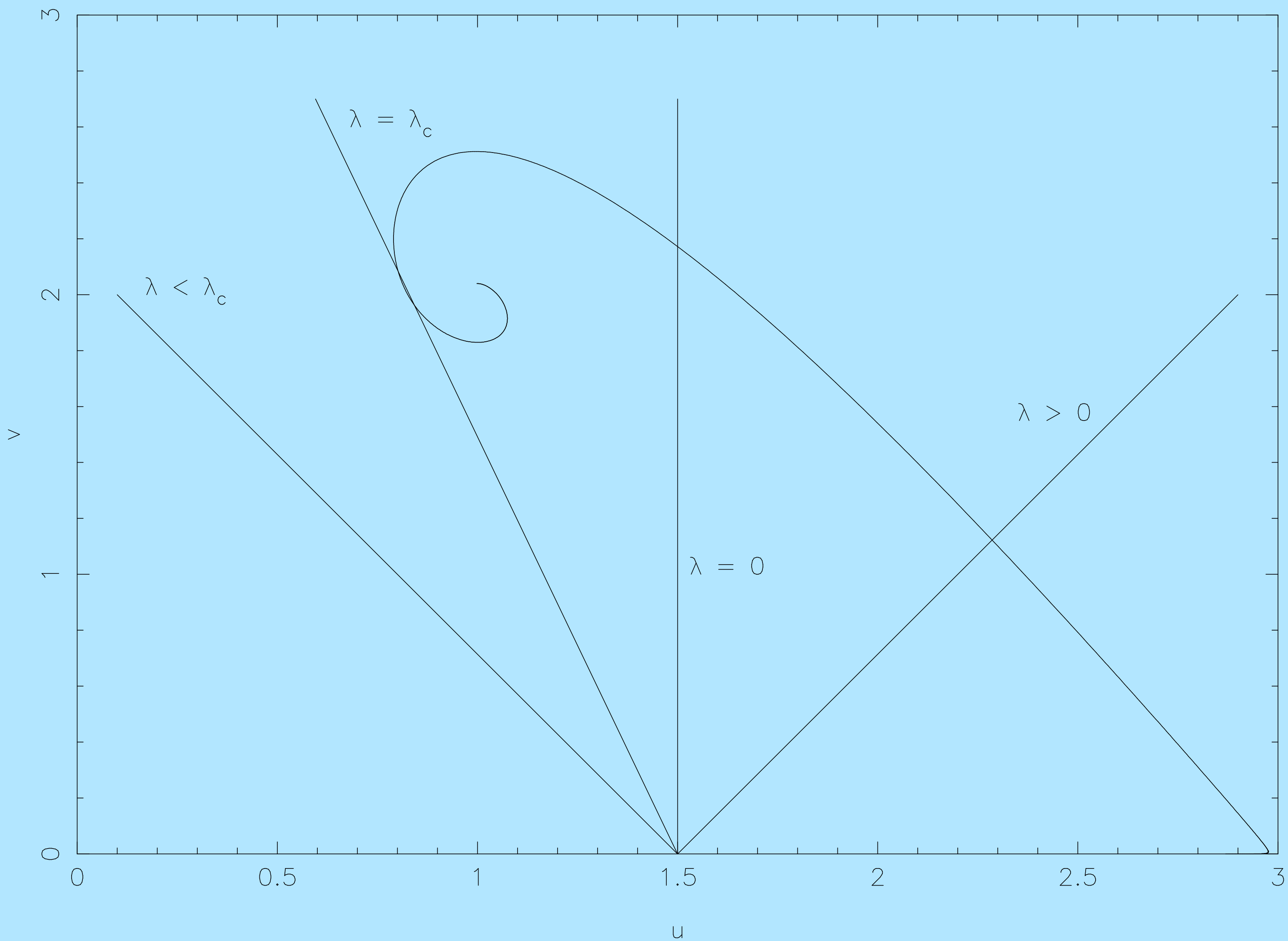
$$\lambda = \frac{RE}{GM^2} = \frac{1}{v_0}\{u_0 - \frac{3}{2}\}.$$

- An isothermal sphere *must* lie on the curve

$$v = \frac{1}{\lambda}\left(u - \frac{3}{2}\right); \qquad \lambda \equiv \frac{RE}{GM^2}$$



[T. Padmanabhan, Physics Reports, **188**, 285 (1990).]

COLLISIONAL RELAXATION IN GRAVITATING SYSTEMS

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$$\langle(\delta v_\perp)^2\rangle_{\rm total} \simeq \Delta t \int_{b_1}^{b_2} (2\pi b db)(vn)\left(\frac{G^2 m^2}{b^2 v^2}\right) = \frac{2\pi n G^2 m^2}{v} \Delta t \ln\left(\frac{b_2}{b_1}\right).$$

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- Take $b_2 = R$= size of the system, $b_1 = b_c$. Then

$$(b_2/b_1) \simeq \left(Rv^2/Gm\right) = N\left(Rv^2/GM\right) \simeq N$$

in virial equilibrium. So

$$t_{soft} \simeq \frac{v^3}{2\pi G^2 m^2 n \ln N} \simeq \left(\frac{N}{\ln N}\right)\left(\frac{R}{v}\right) \simeq \left(\frac{t_{hard}}{\ln N}\right)$$

Publication date: 1942

PRINCIPLES OF STELLAR DYNAMICS

*

By S. CHANDRASEKHAR
Yerkes Observatory

THE UNIVERSITY OF CHICAGO PRESS
CHICAGO · ILLINOIS

Pages 48 to 73 gives the derivation of T_E and T_D!

CHAPTER II

THE TIME OF RELAXATION OF A STELLAR SYSTEM

As we stated in our introduction to the last chapter, in stellar dynamics we are primarily concerned with interpreting the observed state of motions in stellar systems in terms of the forces which govern the motions of the individual stars in the system and the laws of dynamics. In this monograph it will be assumed that the laws of Newtonian dynamics are adequate for such purposes. But this is not to imply that eventually it may not be found necessary to introduce ideas in stellar dynamics which are outside the scope of the classical laws. It is clearly necessary to work out fully the logical consequences of a system of stellar dynamics based on Newtonian laws before we can feel convinced of the need to go outside the framework of such laws. And it is the object of this monograph to set out the general principles of such a classical system of stellar dynamics.

2.1. *An analysis of the nature of the forces acting on a star.*—According to our remarks in the foregoing paragraph, we shall assume that the forces governing the motions of the individual stars in a stellar system are essentially of a gravitational character. In a general way it is clear that these forces arise, first, from the smoothed-out distribution of matter in the system and, second, from the effect of chance stellar encounters. The forces of the first kind are derivable from a gravitational potential $\mathfrak{B}$ representing the smoothed-out distribution of matter in the system. This gravitational potential is a function of the space and time co-ordinates only. On the other hand, the forces of the second kind arise from the accidental encounters with other stars which happen to be in the neighborhood of the star we are considering. More explicitly, the manner in which these two types of forces influence the motion of any particular star can be described as follows: Consider a star which is at the point (x, y, z) at some specified instant of time $t = 0$ (say). Without loss of where $x_0 = jv_2$ and

$$H(x_0) = \frac{1}{2x_0^2}[x_0\Phi'(x_0) + (2x_0^2 - 1)\Phi(x_0)] \qquad (2.431)$$

($\Phi[x_0]$ and $\Phi'[x_0]$ denote, respectively, the error function and its derivative). The function $H(x_0)$ is tabulated in Table 7. Equation

TABLE 7

$H(x_0)$

x_0	$H(x_0)$	T_D/T_E	x_0	$H(x_0)$	T_D/T_E
0.6.	0.421	1.74	1.8.	0.849	0.66
0.8.	.534	1.56	2.0.	.876	.55
1.0.	.629	1.36	2.5.	.920	0.35
1.2.	.706	1.16	3.0.	.944	. . . .
1.4.	.766	0.97	4.0.	0.969	. . . .
1.6.	0.813	0.80			

(2.430) can be written in the form

$$\Sigma \sin^2 2\Psi = \frac{dt}{T_D}, \qquad (2.432)$$

where

$$T_D = \frac{v_2^3}{8\pi NG^2 m_1^2 H(x_0) \log\left[\frac{D_0 v_2^2}{G(m_1 + m_2)}\right]}. \qquad (2.433)$$

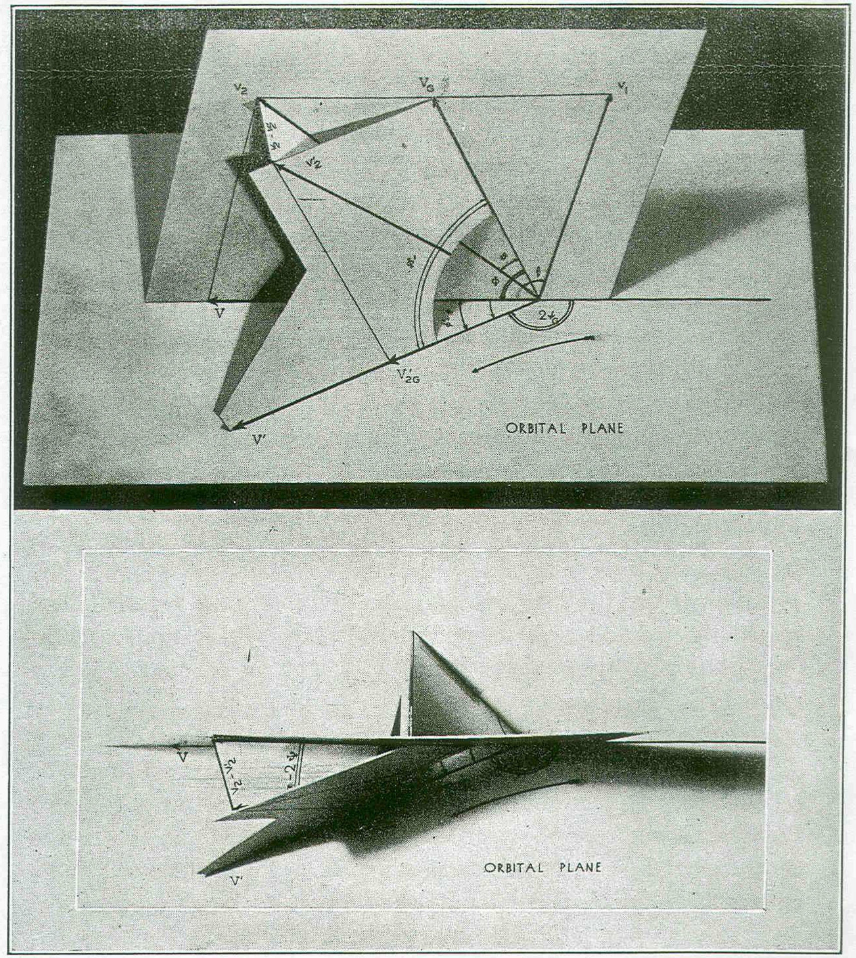

Fig. 13.—Vector model for stellar encounters. The fundamental plane is defined by the vectors v_1 and v_2 representing the velocities of the two stars before the encounter. The velocity of the center of gravity, denoted by V_g, remains constant during the encounter. In a frame of reference in which the center of gravity is at rest, the two stars describe hyperbolae in the orbital plane, which is, in general, inclined at some definite angle to the fundamental plane. The vectors V and v_{2g}, representing respectively the initial relative velocity and the initial velocity of one of the stars with respect to the center of gravity, lie in the orbital plane and are in the same direction. As a result of the encounter, these vectors are deflected by the same angle $\pi - 2\psi_g$ and become respectively V' and v'_{2g}. Finally, $v'_2 = v'_{2g} + V_g$ defines the velocity of the star at the end of the encounter. The angle $\pi - 2\Psi$ between the vectors v_2 and v'_2 measures the true deflection suffered by the star as a result of the encounter (Williamson and Chandrasekhar, *Ap. J.*, **93**, 309, 1941).

ASTRONOMY AND COSMOGONY

BY

SIR JAMES H. JEANS, M.A., D.Sc., LL.D., F.R.S.

SECRETARY OF THE ROYAL SOCIETY, AND RESEARCH
ASSOCIATE OF MOUNT WILSON OBSERVATORY

CAMBRIDGE
AT THE UNIVERSITY PRESS
1929

Pages 317 to 320 contain the derivation by Jeans!

In forming such estimates our unit of time is virtually the interval between one stellar encounter and the next, so that we begin our investigation by considering the frequency of stellar encounters.

THE DYNAMICS OF STELLAR ENCOUNTERS.

285. When two stars come so close as to exert appreciable forces on one another each describes a hyperbolic orbit about the centre of gravity of the two. In fig. 52 let S be the centre of gravity of two stars of masses m, m', which are pulling each other appreciably out of their courses, let V_0 be the velocity of m' before the encounter began, and let V be its velocity at the moment of closest approach, both velocities being measured relative to the centre of gravity S. Let p be the perpendicular distance of the undeflected path from S, and let a be the distance at the instant of closest approach.

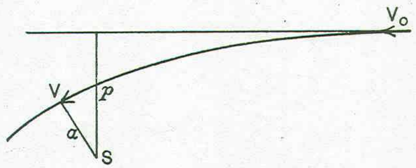

Fig. 52.

The orbit described by m' is that which would be described under a gravitational force $\gamma m^3/(m+m')^2 r^2$ directed towards S. Thus the principles of conservation of energy and momentum supply the relations

$$V^2 - V_0^2 = \frac{2\gamma m^3}{(m+m')^2 a} \quad \text{.....................(285·1)},$$

$$pV_0 = aV \quad \text{.............................(285·2)}.$$

The elimination of V between these equations gives

$$p^2 = a^2 + \frac{2\gamma m^3 a}{(m+m')^2 V_0^2} \quad \text{.....................(285·3)}.$$

The eccentricity of the orbit, e, is given by

$$\frac{e+1}{e-1} = \frac{p^2}{a^2} = 1 + \frac{2\gamma m^3}{(m+m')^2 V_0^2 a},$$

and as total angle of deflection ψ of either orbit is equal to $2 \operatorname{cosec}^{-1} e$, we obtain

$$\frac{\sin \frac{1}{2}\psi}{1 - \sin \frac{1}{2}\psi} = \frac{\gamma m^3}{(m+m')^2 V_0^2 a} \quad \text{..................(285·4)}.$$

This gives the relation between a and ψ; to find the relation between p and ψ we eliminate a between this and equation (285·3) and obtain

$$\tan \tfrac{1}{2}\psi = \frac{\gamma m^3}{(m+m')^2 V_0^2 p} \quad \text{.....................(285·5)}.$$

By differentiation of formula (286·1), we find that there are

$$\frac{\pi\nu\gamma^2 m^6}{(m+m')^4 V_0^3} \frac{\cos\frac{1}{2}\psi}{\sin^3\frac{1}{2}\psi} d\psi$$

encounters in unit time which produce a deflection of path between ψ and $\psi + d\psi$. For small deflections, this may be put in the form

$$\frac{8\pi\nu\gamma^2 m^6}{(m+m')^4 V_0^3} \frac{d\psi}{\psi^3} \quad \text{.......................(287·1).}$$

The cumulative effect of encounters which produce small deflections $\psi_1, \psi_2, \ldots$ is to produce a deflection of which the expectation Ψ is given by

$$\Psi^2 = \psi_1^2 + \psi_2^2 + \ldots \quad \text{.......................(287·2).}$$

Let $\psi_1, \psi_2, \ldots$ be all the deflections of amount between two limits α and β which occur within a time t. Then, from formula (287·1), we find that

$$\begin{aligned}\Psi^2 &= t\int_\alpha^\beta \frac{8\pi\nu\gamma^2 m^6}{(m+m')^4 V_0^3} \frac{d\psi}{\psi} \\ &= \frac{8\pi\nu\gamma^2 m^6}{(m+m')^4 V_0^3}\, t \log_e\left(\frac{\beta}{\alpha}\right) \quad \text{.................(287·3).}\end{aligned}$$

Let us take the upper limit of deflection to be $\beta = \frac{1}{2}\pi$, thus considering the cumulative effect of deflections less than those considered in § 286. It might at first be thought that to take account of all deflections of amount less than $\frac{1}{2}\pi$, we ought to take $\alpha = 0$, but such a procedure would be erroneous for the following reason.

Formula (287·2) is only accurate if the deflections $\psi_1, \psi_2, \ldots$ are independent, and this requires that they should originate in distinct encounters. If ψ is allowed to become very small, the corresponding distance a of closest approach, as given by equation (285·4), becomes very large, so that there are several stars within a distance a at the same instant, and their effects tend to neutralise one another. To obtain correct results we must stop off the integration before it brings us to values of a as large as this.

We must accordingly choose the lower limit α so as to correspond to a distance of closest approach which is about equal to the distance between adjacent stars, and so to $\nu^{-\frac{1}{3}}$. By equation (285·4), this value of α is given by

$$\alpha = \frac{2\gamma m^3 \nu^{\frac{1}{3}}}{(m+m')^2 V_0^{\,2}}.$$

Assigning this value to α and putting $\beta = \frac{1}{2}\pi$, equation (287·3) becomes

$$\Psi^2 = \frac{8\pi\nu\gamma^2 m^6}{(m+m')^4 V_0^{\,3}} \log_e \left(\frac{\pi (m+m')^2 V_0^{\,2}}{4\gamma m^3 \nu^{\frac{1}{3}}}\right) t \quad \text{.........(287·4)},$$

or, inserting the numerical values already mentioned,

$$\Psi^2 = \frac{8\pi\nu\gamma^2 m^6}{(m+m')^4 V_0^{\,3}} \times 11{\cdot}9t.$$

The time necessary for deflections less than a right angle to produce a resultant deflection equal to a right angle is obtained by putting $\Psi = \frac{1}{2}\pi$, and is found to be

$$t = 0{\cdot}026 \frac{(m+m')^4 V_0^{\,3}}{\pi\nu\gamma^2 m^6} \text{ seconds} \quad \text{..............(287·5)}.$$

Comparing with formula (286·3) we see that this time is only about one-fortieth of that needed for a single encounter to deflect the path by a right angle. With the values already used it is equal to about 5×10^{13} years.

First appearance of $\ln N$

ON THE DYNAMICS OF OPEN CLUSTERS

(orig.: Uch. Zap. L.G.U. No. 22, p. 19; 1938)

V. A. Ambartsumian

It has already been pointed out in the literature that due to several causes, open star clusters dissipate with time. For instance, Rosseland showed that when external stars move through a cluster, they cause a perturbation of the motion of the stars in the cluster and could transfer enough momentum to individual stars to cause their escape from the cluster's gravitational field. In this way the cluster will lose stars gradually, i.e., it will dissipate. According to Rosseland the time needed for the star cluster to dissipate following the outlined mechanism is 10^{10} years. However, as pointed out by the author of this article in the supplement to the Russian edition of Rosseland's book, there is another factor that makes the life of the open cluster even shorter: the stars in the cluster have close encounters with each other, as a result of which they exchange kinetic energy and gradually tend towards the most probable distribution, i.e., a Maxwell-Boltzmann distribution. And this, as we shall see shortly, also causes the dissipation of the cluster.

The relaxation time, i.e. the time in which the encounters of the stars in the cluster will lead to statistical equilibrium, is given approximately by the formula:

$$\tau = \frac{3\sqrt{2}}{32\pi n} \frac{v^3}{G^2 m^2 \ln\left(\frac{\rho}{\rho_0}\right)} , \qquad (1)$$

On the other hand

$$2T = Nmv^2 \quad .$$

Therefore the virial theorem assumes the form:

$$v^2 = \frac{GNm}{2\rho} \quad . \tag{4}$$

Comparing (4) with (2), we find that

$$\ln\left(\frac{\rho}{\rho_0}\right) = \ln\left(\frac{N}{4}\right); \tag{5}$$

substituting (4) and (5) in (1) and taking into account that

$$n = \frac{N}{\frac{4}{3}\pi\rho^3} \quad ,$$

CHANDRA'S COMMENT ON THE WORK OF JEANS

Though the physical ideas were correctly formulated by Jeans and Schwarzschild, a completely rigorous evaluation of the time of relaxation was not available until recently. The analysis in §§ 2.3 and 2.4 are, in the main, taken from—

6. S. CHANDRASEKHAR, *Ap. J.*, **93**, 285, 1941, and—

SECOND ASPECT OF DIFFUSION IN VELOCITY SPACE

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SECOND ASPECT OF DIFFUSION IN VELOCITY SPACE

- This can't be the whole story!
- We need a dynamical friction to reach steady state with Maxwellian distribution of velocities.
- Chandra seems to have realized this soon after the publication of the book!!

NEW METHODS IN STELLAR DYNAMICS*

By

S. Chandrasekhar †

* Awarded an A. Cressy Morrison Prize in Natural Science in 1942 by The New York Academy of Sciences. Publication made possible through a grant from the income of the Ralph Winifred Tower Memorial Fund.

† Yerkes Observatory, Williams Bay, Wisconsin.

$$- \frac{2}{3} \pi G \overline{M} n B \left(\frac{|F|}{Q_H} \right) \left(v - 3 \frac{v \cdot F}{|F|^2} F \right) \tag{54}$$

along any particular direction gives the average value of the rate of change in the force $\boldsymbol{F}$ per unit mass acting on a star that is to be expected in the specified direction, when the star is moving with a velocity $\boldsymbol{v}$ in an appropriately chosen local standard of rest. Stated in this manner, we at once see the essential difference in the stochastic variations of $\boldsymbol{F}$ with time in the two cases $|\boldsymbol{v}| = 0$ and $|\boldsymbol{v}| \neq 0$. In the former case, $\bar{\dot{\boldsymbol{F}}} \equiv 0$; but this is not generally true when $|\boldsymbol{v}| \neq 0$. Or expressed differently, when $|\boldsymbol{v}| = 0$ the changes in $\boldsymbol{F}$ occur with equal probability in all directions, while this is not the case when $|\boldsymbol{v}| \neq 0$. The true nature of this difference is brought out very clearly when we consider

$$\overline{\left(\frac{d|\boldsymbol{F}|}{dt} \right)}_{\boldsymbol{F}, \boldsymbol{v}} \tag{55}$$

according to equation (49). Remembering that $B(\beta) \geq 0$ for $\beta \geq 0$, we conclude from equation (49) that

$$\overline{\left(\frac{d|\boldsymbol{F}|}{dt} \right)}_{\boldsymbol{F}, \boldsymbol{v}} > 0 \quad \text{if} \quad \boldsymbol{v} \cdot \boldsymbol{F} > 0, \tag{56}$$

and

$$\overline{\left(\frac{d|\boldsymbol{F}|}{dt} \right)}_{\boldsymbol{F}, \boldsymbol{v}} < 0 \quad \text{if} \quad \boldsymbol{v} \cdot \boldsymbol{F} < 0. \tag{57}$$

In other words, if $\boldsymbol{F}$ has a positive component in the direction of $\boldsymbol{v}$, $|\boldsymbol{F}|$ increases on the average, while if $\boldsymbol{F}$ has a negative component in the direction of $\boldsymbol{v}$, $|\boldsymbol{F}|$ decreases on the average. This essential asymmetry introduced by the direction of $\boldsymbol{v}$ may be expected to give rise to the phenomenon of dynamical friction.

REVIEWS OF MODERN PHYSICS VOLUME 21, NUMBER 3 JULY, 1949

Brownian Motion, Dynamical Friction, and Stellar Dynamics

S. CHANDRASEKHAR
Yerkes Observatory, University of Chicago, Williams Bay, Wisconsin

loses all trace of its initial state as time progresses. Such a gradual loss of "memory" can be achieved only by the operation of a dissipative force like dynamical friction which will gradually damp out any given initial velocity. Thus, if we assume for the sake of simplicity, that η is independent of $|\mathbf{u}|$, then the average velocity at later times will tend to zero like

$$\bar{\mathbf{u}}=\mathbf{u}_0 e^{-\eta t}; \tag{38}$$

but this is not to imply that the mean square velocity also tends to zero. Indeed, the restoration of a Maxwellian distribution of velocities from an arbitrary initial state requires that

$$\bar{\mathbf{u}}\rightarrow 0 \text{ while } \langle|\mathbf{u}|^2\rangle_{Av}\rightarrow \text{a constant as } t\rightarrow\infty. \tag{39}$$

To achieve the first of these conditions we need dynamical friction and to achieve the second we need random fluctuations as expressed by a diffusion coefficient. The recognition of these facts is, of course, Einstein's achievement.

DIFFUSION IN VELOCITY SPACE:
A UNIFIED APPROACH

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A UNIFIED APPROACH

- Diffusion current as the source term:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} - \nabla\phi.\frac{\partial f}{\partial \mathbf{v}} = -\frac{\partial J^\alpha}{\partial p^\alpha}$$

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- Form of the current can be shown to be:

$$J_\alpha = \frac{B_0}{2} \int d\,\mathbf{l}' \left\{ f \frac{\partial f'}{\partial l_\beta} - f' \frac{\partial f}{\partial l_\beta} \right\} . \left\{ \frac{\delta_{\alpha\beta}}{k} - \frac{k_\alpha k_\beta}{k^3} \right\}$$

where $B_0 = 4\pi G^2 m^5 L$; and $L = \int_{b_1}^{b_2} \frac{db}{b} = \ln\left(\frac{b_2}{b_1}\right)$

TWO FOR THE PRICE OF ONE!

- Current:

$$J_\alpha = \frac{B_0}{2} \int d\,\mathbf{l}' \left\{ f \frac{\partial f'}{\partial l_\beta} - f' \frac{\partial f}{\partial l_\beta} \right\} \cdot \left\{ \frac{\delta_{\alpha\beta}}{k} - \frac{k_\alpha k_\beta}{k^3} \right\}$$

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- The term proportional to f gives dynamical friction. The term proportional to $(\partial f / \partial l_\beta)$ increases the velocity dispersion.

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- The current J_α vanishes for Maxwell distribution, as it should!

AN ILLUSTRATIVE TOY MODEL

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- Note that

$$J_\alpha(l) \equiv a_\alpha(\mathbf{l}) f(\mathbf{l}) - \frac{1}{2}\frac{\partial}{\partial l_\beta}\left\{\sigma^2_{\alpha\beta} f\right\}$$

where $a_\alpha = (\partial\eta/\partial l_\alpha)\,, \sigma^2_{\alpha\beta} = (\partial^2\psi/\partial l_\alpha \partial l_\beta)$

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- Aside:

$$\nabla^2\psi = \eta; \qquad \nabla^2_l \eta(\mathbf{l}) = \nabla^2_l \left\{2\int d\mathbf{l}' \frac{f(\mathbf{l}')}{|\mathbf{l}-\mathbf{l}'|}\right\} = -8\pi f(\mathbf{l})$$

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- Treat the coefficients as constants to understand the structure of the equation:

$$\frac{\partial f(v,t)}{\partial t} = \frac{\partial}{\partial v}\left\{(\alpha v) f + \frac{\sigma^2}{2}\frac{\partial f}{\partial v}\right\} \equiv -\frac{\partial J}{\partial v}$$

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- An initial distribution $f(v,0) = \delta_D(v - v_0)$ evolves to:

$$f(v,t) = \left[\frac{\alpha}{\pi\sigma^2(1-\mathrm{e}^{-2\alpha t})}\right]^{1/2} \exp\left[-\frac{\alpha(v - v_0\mathrm{e}^{-\alpha t})^2}{\sigma^2(1-\mathrm{e}^{-2\alpha t})}\right]$$

- The mean velocity decays to zero:

$$< v >= v_0 \mathrm{e}^{-\alpha t}$$

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$$< v > = v_0 \mathrm{e}^{-\alpha t}$$

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- The interplay between the two effects is obvious.

HISTORY: OVERLOOKING LANDAU (1936)!

DIE KINETISCHE GLEICHUNG FÜR DEN FALL COULOMBSCHER WECHSELWIRKUNG.

Von L. Landau.

(Eingegangen am [illegible]. Juni 1936.)

Es wird die kinetische Gleichung für die aus geladenen Teilchen bestehenden Systeme unter Berücksichtigung ihrer Wechselwirkung abgeleitet. Die Grössenordnung der freien Weglänge dieser Teilchen wird bestimmt, sowie die Geschwindigkeit des Temperaturausgleiches von Ionen und Elektronen im Plasma.

Wenn wir diesen Ausdruck in (3) einsetzen, finden wir für den Strom der Teilchen e im Impulsraum endgültig:

$$j_i = \pi e^4 L \Sigma e'^2 \int \left\{ n \frac{\partial n'}{\partial p_k} - n' \frac{\partial n}{\partial p_k} \right\} \frac{u^2 \delta_{ik} - u_i u_k}{u^3} \, d\tau'. \quad (7)$$

HISTORY: OVERLOOKING LANDAU (1936)!

PHYSICAL REVIEW VOLUME 80, NUMBER 2 OCTOBER 15, 1950

The Electrical Conductivity of an Ionized Gas*

ROBERT S. COHEN**
Sloane Physics Laboratory, Yale University, New Haven, Connecticut

AND

LYMAN SPITZER, JR., AND PAUL McR. ROUTLY
Princeton University Observatory, Princeton, New Jersey

(Received April 3, 1950)

bution function is affected primarily by the many small deflections produced by relatively distant encounters. There will be many such encounters during the time a particle travels over its mean free path, and the change in the particle velocity can be computed in the same way as is the change of the position of a particle in

* This work has been supported in part by the ONR.

** National Research Council Predoctoral Fellow (1946–48); now with the Department of Philosophy, Yale University. This material was submitted in part to Yale University in partial fulfillment of the requirements for a Ph.D. degree.

[1] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1939).

[2] T. G. Cowling, Proc. Roy. Soc. **A**, **183**, 453 (1945).

[3] R. Landshoff, Phys. Rev. **76**, 904 (1949).

[4] S. Chandrasekhar, Astrophys. J. **97**, 255, 263 (1943).

I. GENERAL PRINCIPLES

The velocity distribution function f_r for particles of type r, interacting with particles of different types s, is determined by Boltzmann's equation (reference 1, Eq. (8.1_1))

$$\frac{\partial f_r}{\partial t}+\sum_i v_{ri}\frac{\partial f_r}{\partial x_i}+\sum_i F_{ri}\frac{\partial f_r}{\partial v_{ri}}=\sum_s\left(\frac{\partial_e f_r}{\partial t}\right)_s, \qquad (1)$$

where the notation is similar to that used by Chapman

[5] L. Landau, Physik Zeits. Sowjetunion **10**, 154 (1936). In this reference, the important terms representing dynamical friction, which should appear in the diffusion equation, are set equal to zero as a result of certain approximations.

Footnote is incorrect; Landau's expression has both dynamical friction and velocity dispersion!

THE
PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 107, NO. 1 JULY 1, 1957

Fokker-Planck Equation for an Inverse-Square Force*

MARSHALL N. ROSENBLUTH,† WILLIAM M. MACDONALD,†† AND DAVID L. JUDD
Radiation Laboratory, University of California, Berkeley, California
(Received August 31, 1956)

malism of this equation to evaluate the collision terms of the Boltzmann equation under the assumptions that (a) the events producing changes in particle momenta

* This work was done under the auspices of the U. S. Atomic Energy Commission.
† Present address: General Atomic, San Diego, California.
†† Present address: Physics Department, University of Maryland, College Park, Maryland.
[1] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1952), second edition, pp. 178–179.
[2] S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

II. FORMULATION OF THE PROBLEM

The Boltzmann equation for the change of the molecular distribution function is given by

$$\frac{\partial f_a}{\partial t}+v^\mu\frac{\partial f_a}{\partial x^\mu}+\frac{F^\mu}{m}\frac{\partial f_a}{\partial v^\mu}=\left(\frac{\partial f_a}{\partial t}\right)_c, \qquad (1)$$

[3] Cohen, Spitzer, and McRoutly, Phys. Rev. **80**, 230 (1950). A more complete list of references is given in this paper.
[4] L. Spitzer and R. Harm, Phys. Rev. **89**, 977 (1953).

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CHICAGO · ILLINOIS 60637
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*Laboratory for Astrophysics
and Space Research*

September 12, 1989

Dr. T. Padmanabhan
Theoretical Astrophysics Group
Tata Institute of Fundamental Research
Homi Bhabha Road
BOMBAY 400 005, INDIA

Dear Dr. Padmanabhan,

Thank you for your letter of August 23 enclosing a copy of your preprint "Statistical mechanics of gravitating systems." While I appreciate your courtesy in sending me this preprint, it relates to matters that I was interested in, some forty and more years ago. And I am afraid that my present remembrance is faded.

It is possible that I may visit the Inter-University Center for Astronomy and Astrophysics in Poona on a date in December (not certain yet). Perhaps I may have a chance to see you on that occasion.

Yours sincerely,

S. Chandrasekhar

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GRAVITATIONAL CLUSTERING IN EXPANDING UNIVERSE

– SOME KEY ISSUES –

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- Do the virialized structures formed in an expanding universe due to gravitational clustering have any invariant properties? Can their structure be understood from first principles?

- How can one connect up the local behaviour of gravitating systems to the evolution of clustering in the universe?

BASIC DEFINITIONS

- Density:

$$\rho(\mathbf{x},t) = \frac{m}{a^3(t)} \sum_i \delta_D[\mathbf{x} - \mathbf{x}_i(t)]$$

- Mean density:

$$\rho_b(t) \equiv \int \frac{d^3\mathbf{x}}{V} \rho(\mathbf{x},t) = \frac{m}{a^3(t)} \left(\frac{N}{V}\right) = \frac{M}{a^3 V} = \frac{\rho_0}{a^3}$$

- Density contrast:

$$1 + \delta(\mathbf{x},t) \equiv \frac{\rho(\mathbf{x},t)}{\rho_b} = \frac{V}{N} \sum_i \delta_D[\mathbf{x} - \mathbf{x}_i(t)] = \int d\mathbf{q} \delta_D[\mathbf{x} - \mathbf{x}_T(t,\mathbf{q})].$$

- Density contrast in Fourier space:

$$\delta_{\mathbf{k}}(t) \equiv \int d^3\mathbf{x} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \delta(\mathbf{x},t) = \int d^3\mathbf{q}\ \exp[-i\mathbf{k}.\mathbf{x}_T(t,\mathbf{q})] - (2\pi)^3 \delta_D(\mathbf{k})$$

THE EXACT (BUT USELESS) DESCRIPTION

- Density contrast in Fourier space satisfies:

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} = 4\pi G\rho_b\delta_{\mathbf{k}} + A_{\mathbf{k}} - B_{\mathbf{k}}$$

with

$$A_{\mathbf{k}} = 4\pi G\rho_b \int \frac{d^3\mathbf{k}'}{(2\pi)^3}\delta_{\mathbf{k}'}\delta_{\mathbf{k}-\mathbf{k}'}\left[\frac{\mathbf{k}.\mathbf{k}'}{k'^2}\right]$$

$$B_{\mathbf{k}} = \int d^3\mathbf{q}\,(\mathbf{k}.\dot{\mathbf{x}}_T)^2 \exp\left[-i\mathbf{k}.\mathbf{x}_T(t,\mathbf{q})\right]$$

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$$B_{\mathbf{k}} = \int d^3\mathbf{q}\,(\mathbf{k}.\dot{\mathbf{x}}_T)^2 \exp\left[-i\mathbf{k}.\mathbf{x}_T(t,\mathbf{q})\right]$$

- Coupled exact equations:

$$\begin{aligned}\ddot{\phi}_{\mathbf{k}} + 4\frac{\dot{a}}{a}\dot{\phi}_{\mathbf{k}} &= -\frac{1}{2a^2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}}\phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}}\left[\left(\frac{k}{2}\right)^2 + p^2 - 2\left(\frac{\mathbf{k}.\mathbf{p}}{k}\right)^2\right] \\ &+ \left(\frac{3H_0^2}{2}\right)\int \frac{d^3\mathbf{q}}{a}\left(\frac{\mathbf{k}.\dot{\mathbf{x}}}{k}\right)^2 e^{i\mathbf{k}.\mathbf{x}}\end{aligned}$$

$$\ddot{\mathbf{x}} + 2\frac{\dot{a}}{a}\dot{\mathbf{x}} = -\frac{1}{a^2}\nabla_x\phi = -\frac{1}{a^2}\int i\mathbf{k}\phi_{\mathbf{k}}\exp i(\mathbf{k}\cdot\mathbf{x});$$

“Renormalizability” of gravity

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- One can then show that: The term $(A_{\mathbf{k}} - B_{\mathbf{k}})$ receives contribution only from particles which are not bound to any of the clusters to the order $\mathcal{O}(k^2 R^2)$. (Peebles, 1980)

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- Allows one to ignore contributions from virialised systems and treat the rest in Zeldovich (-like) approximation. Then one gets:

$$H_0^2\left(a\frac{d^2}{da^2}+\frac{7}{2}\frac{d}{da}\right)\phi_{\mathbf{k}} \;=\; -\frac{2}{3}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\phi^L_{\frac{1}{2}\mathbf{k}+\mathbf{p}}\phi^L_{\frac{1}{2}\mathbf{k}-\mathbf{p}}\left[\left(\frac{k}{2}\right)^2-\left(\frac{\mathbf{k}.\mathbf{p}}{k}\right)^2\right]$$
$$-\frac{1}{2}\int'\frac{d^3\mathbf{p}}{(2\pi)^3}\phi_{\frac{1}{2}\mathbf{k}+\mathbf{p}}\phi_{\frac{1}{2}\mathbf{k}-\mathbf{p}}\left[\left(\frac{k}{2}\right)^2+p^2-2\left(\frac{\mathbf{k}.\mathbf{p}}{k}\right)^2\right]$$

(T.P, 2002)

Evolution at large scales

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- Ignore the terms $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$. Then:

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- For $a \propto t^{2/3}, \rho_b \propto a^{-3}$, the growing solution is:

$$\delta_k \propto a; \quad P(k) = |\delta_k|^2 \propto a^2; \quad \xi(a, x) \propto a^2$$

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- BUT: If $\delta_{\mathbf{k}} \to 0$ for certain range of $\mathbf{k}$ at $t = t_0$ (but is nonzero elsewhere) then $(A_{\mathbf{k}} - B_{\mathbf{k}}) \gg 4\pi G\rho_b\delta_{\mathbf{k}}$ and the growth of perturbations around $\mathbf{k}$ will be entirely determined by nonlinear effects.

Evolution at large scales

- Ignore the terms $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$. Then:

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} = 4\pi G\rho_b\delta_{\mathbf{k}}$$

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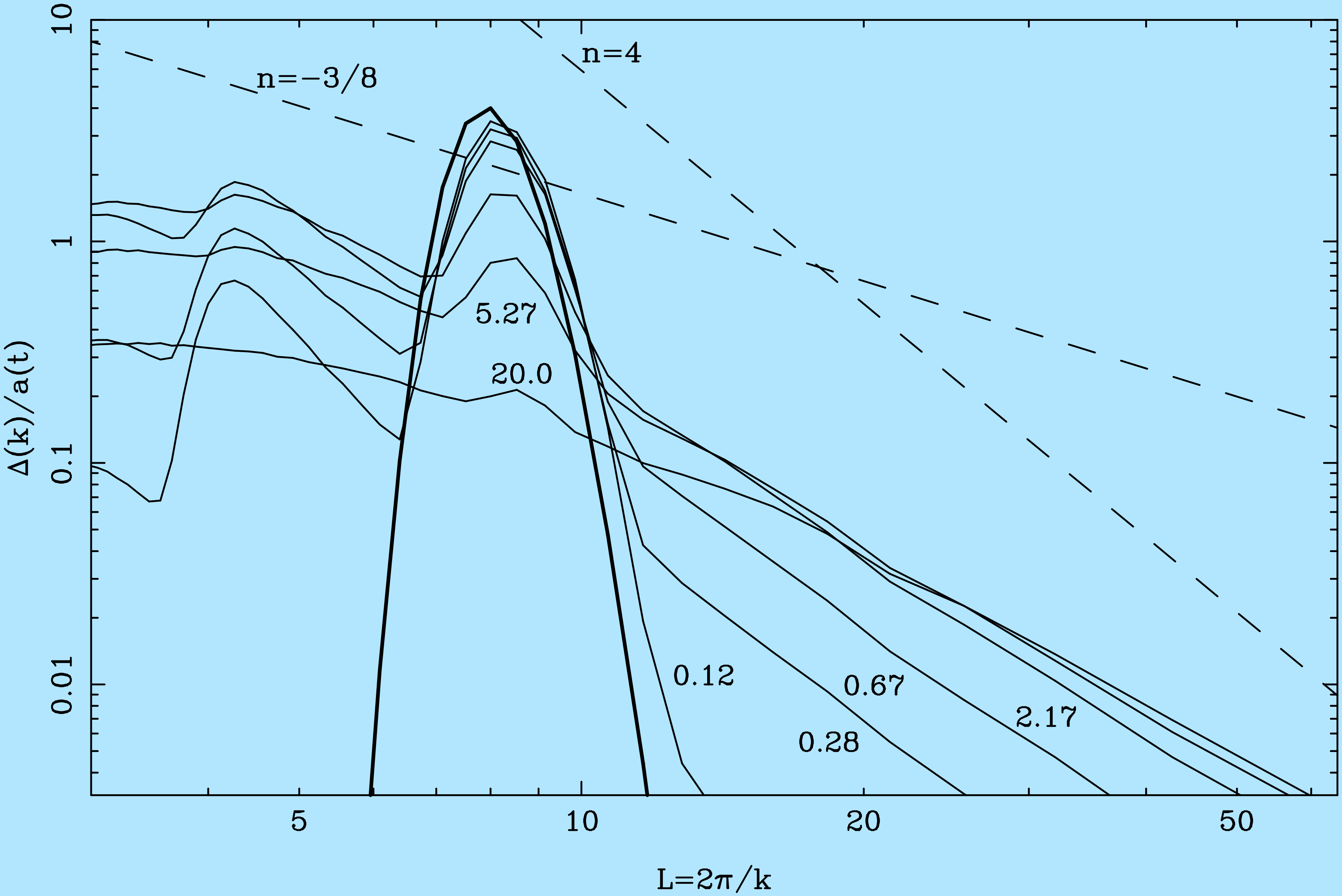
- There is inverse cascade of power in gravitational clustering! (Zeldovich, 1965)



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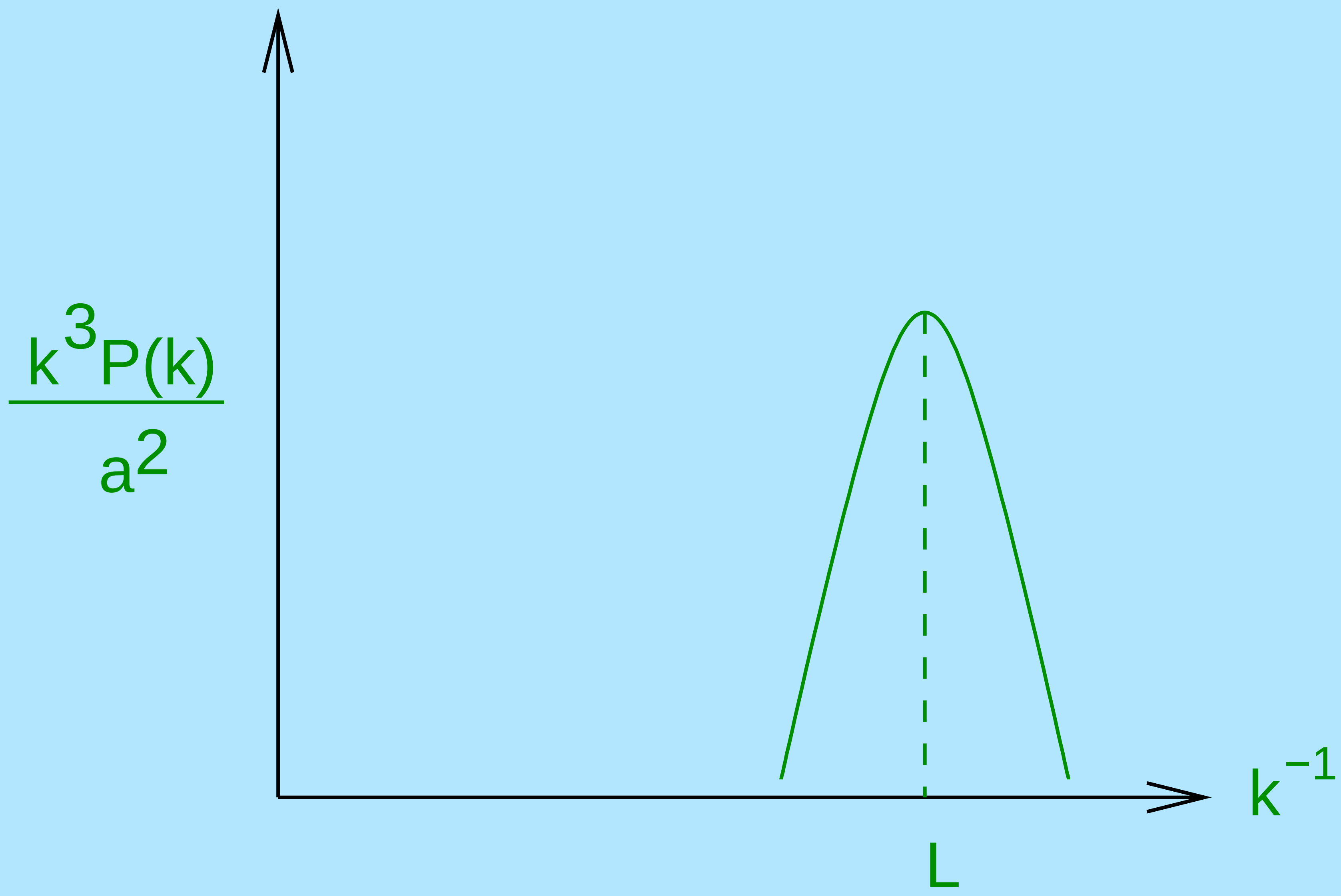
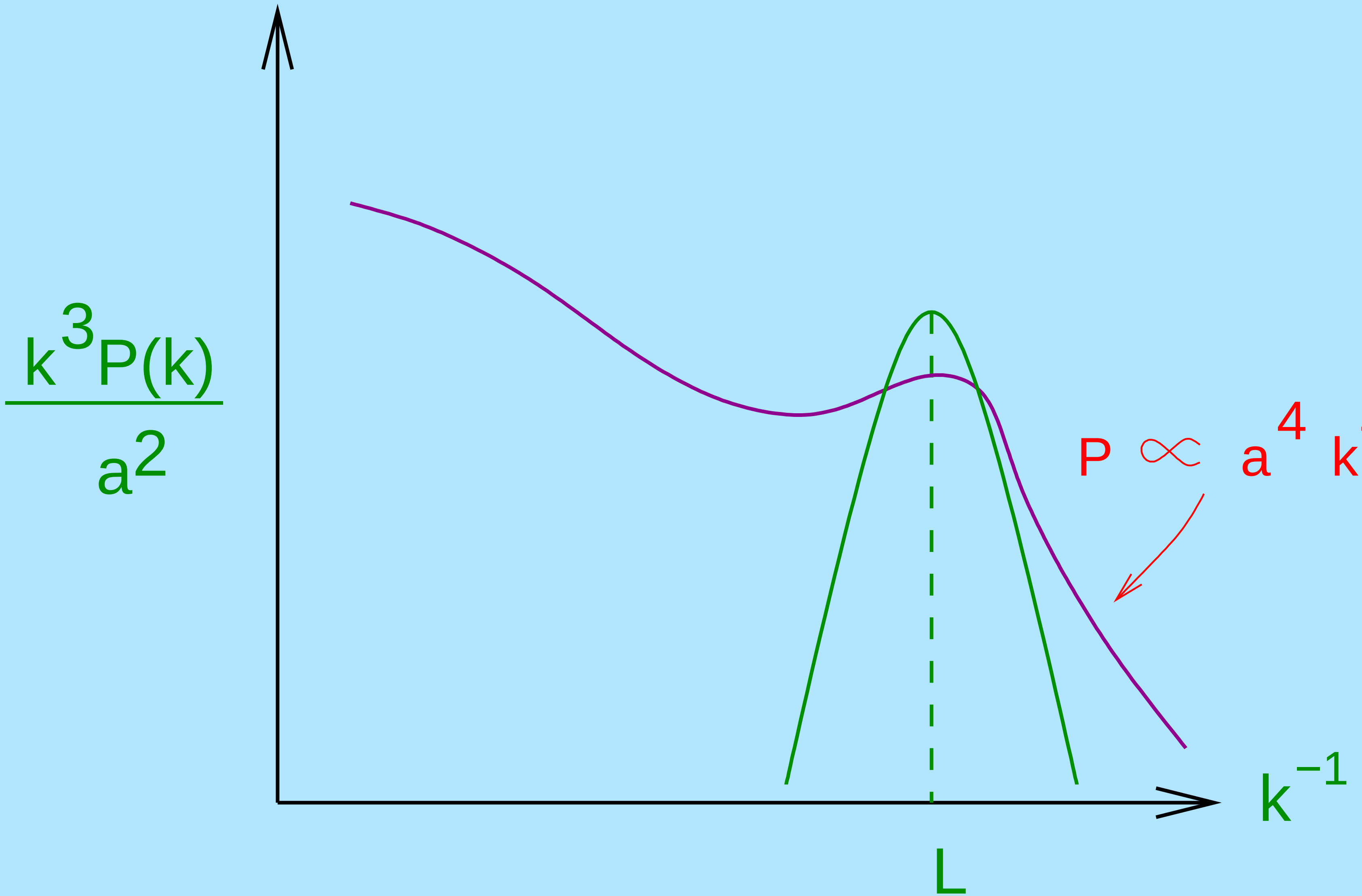
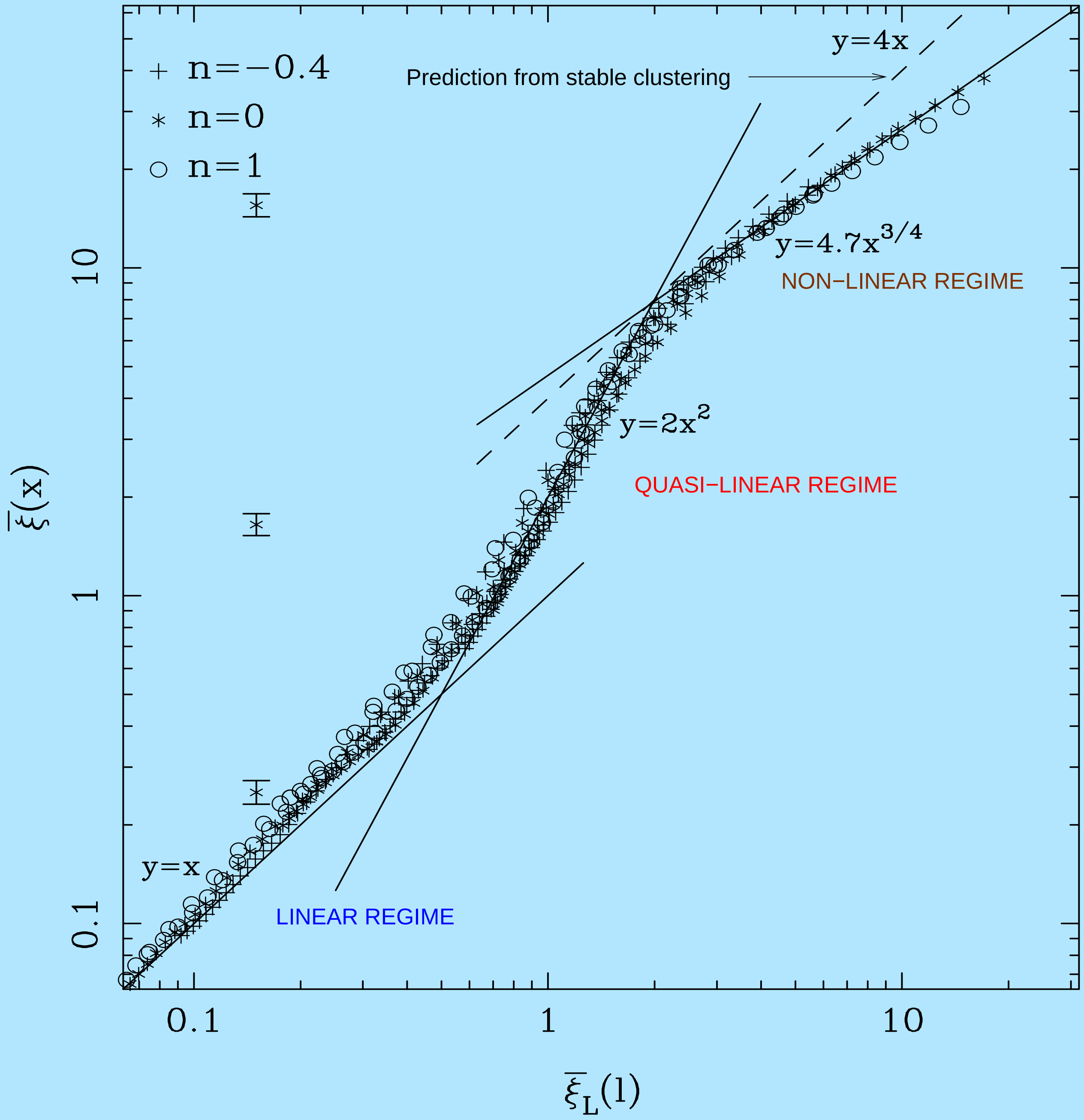


$\frac{k^3 P(k)}{a^2}$
k^{-1}
L


$\frac{k^3 P(k)}{a^2}$
$P \propto a^4 k^4$
k^{-1}
L



Bagla, Engineer, TP (1998), Ap.J.,**495**, 25

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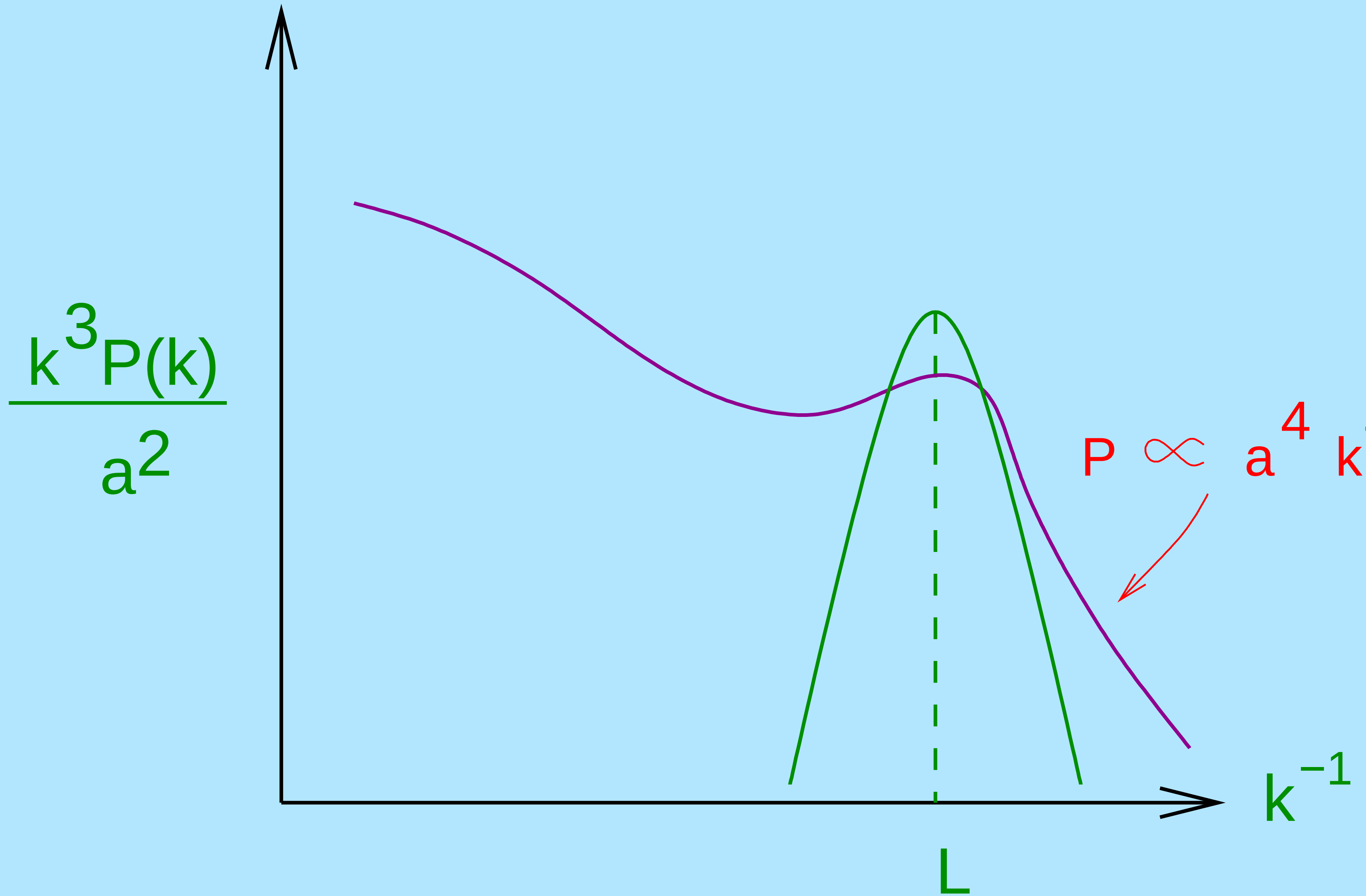
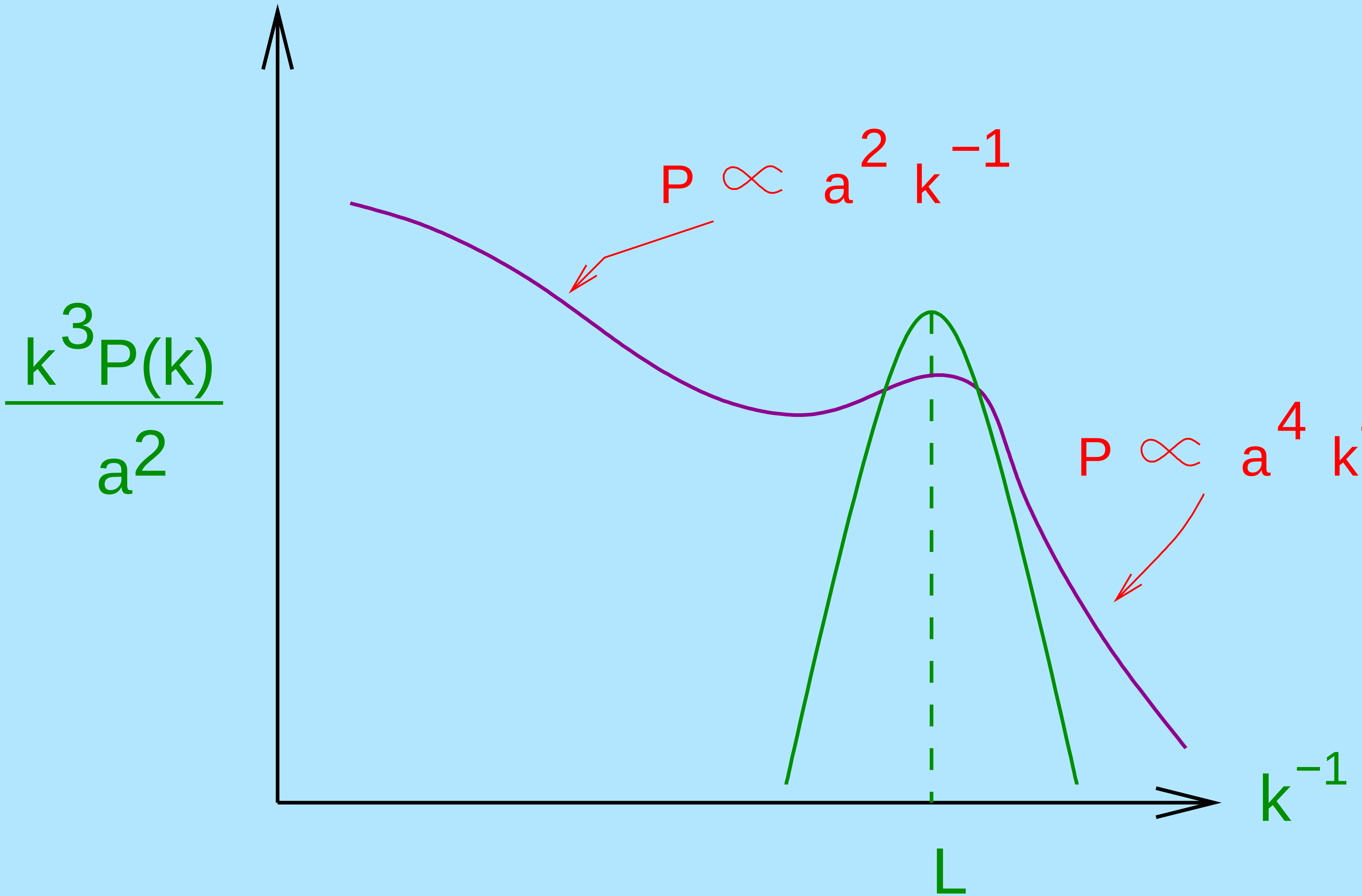
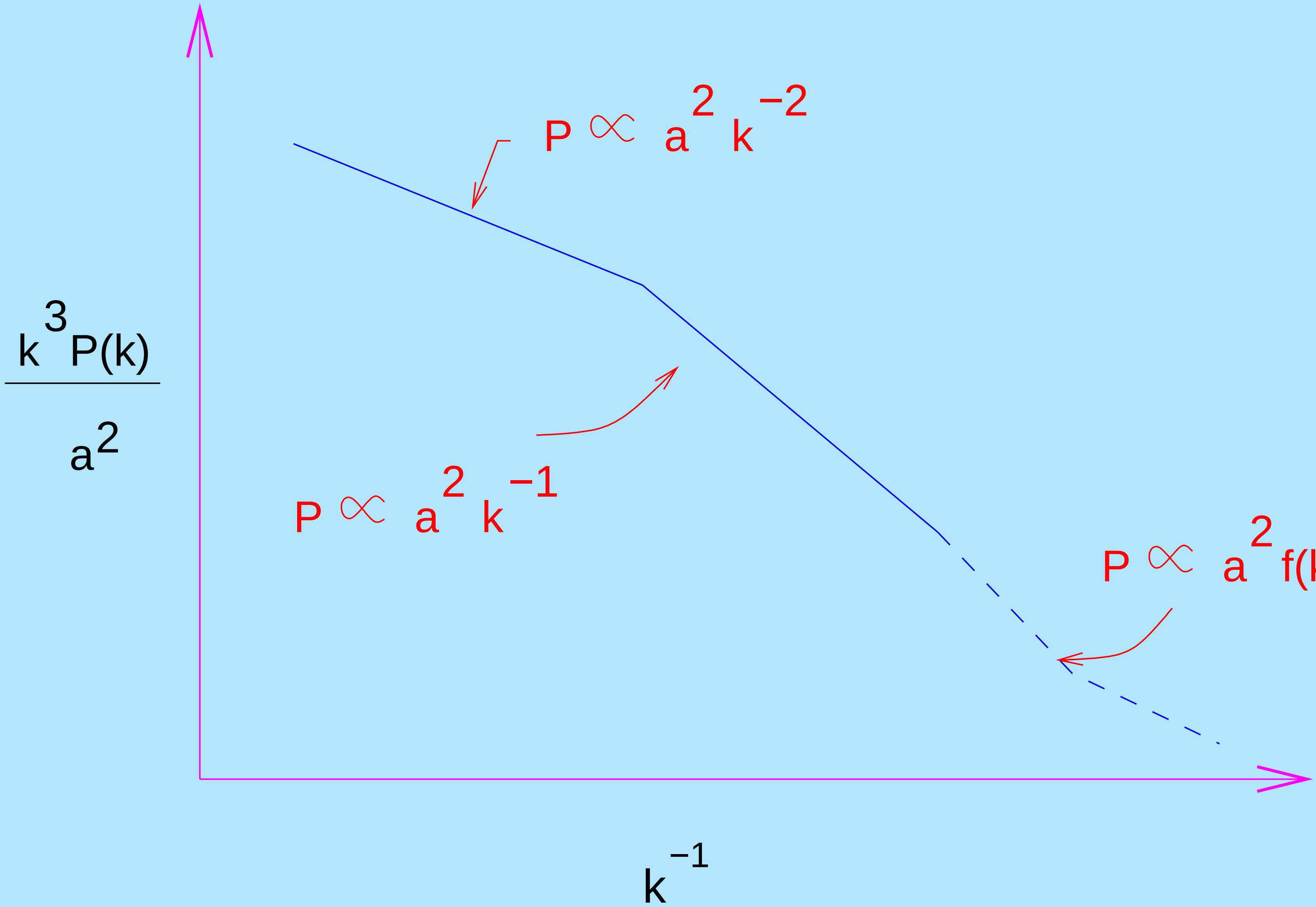
- NEW FEATURE: The energy flow is form invariant (“equipartition”) when $n=-1$ in QL and $n=-2$ in the NL regimes! Then $E \propto a$ in all three regimes.

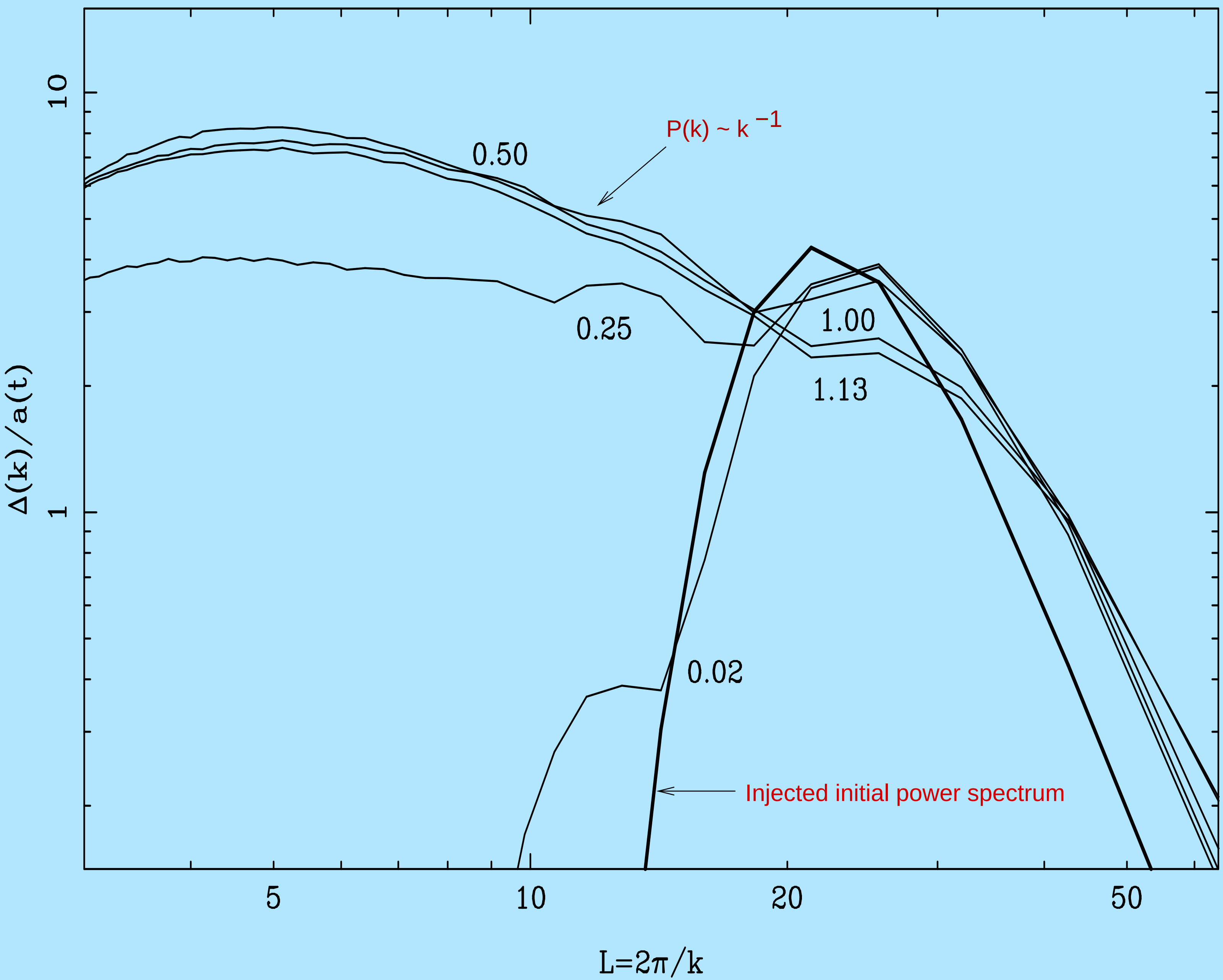


$\frac{k^3 P(k)}{a^2}$
$P \propto a^4 k^4$
k^{-1}
L



$P \propto a^2 k^{-1}$
$\frac{k^3 P(k)}{a^2}$
$P \propto a^4 k^4$
k^{-1}
L


$P \propto a^2 k^{-2}$
$\frac{k^3 P(k)}{a^2}$
$P \propto a^2 k^{-1}$
$P \propto a^2 f(k)$
k^{-1}



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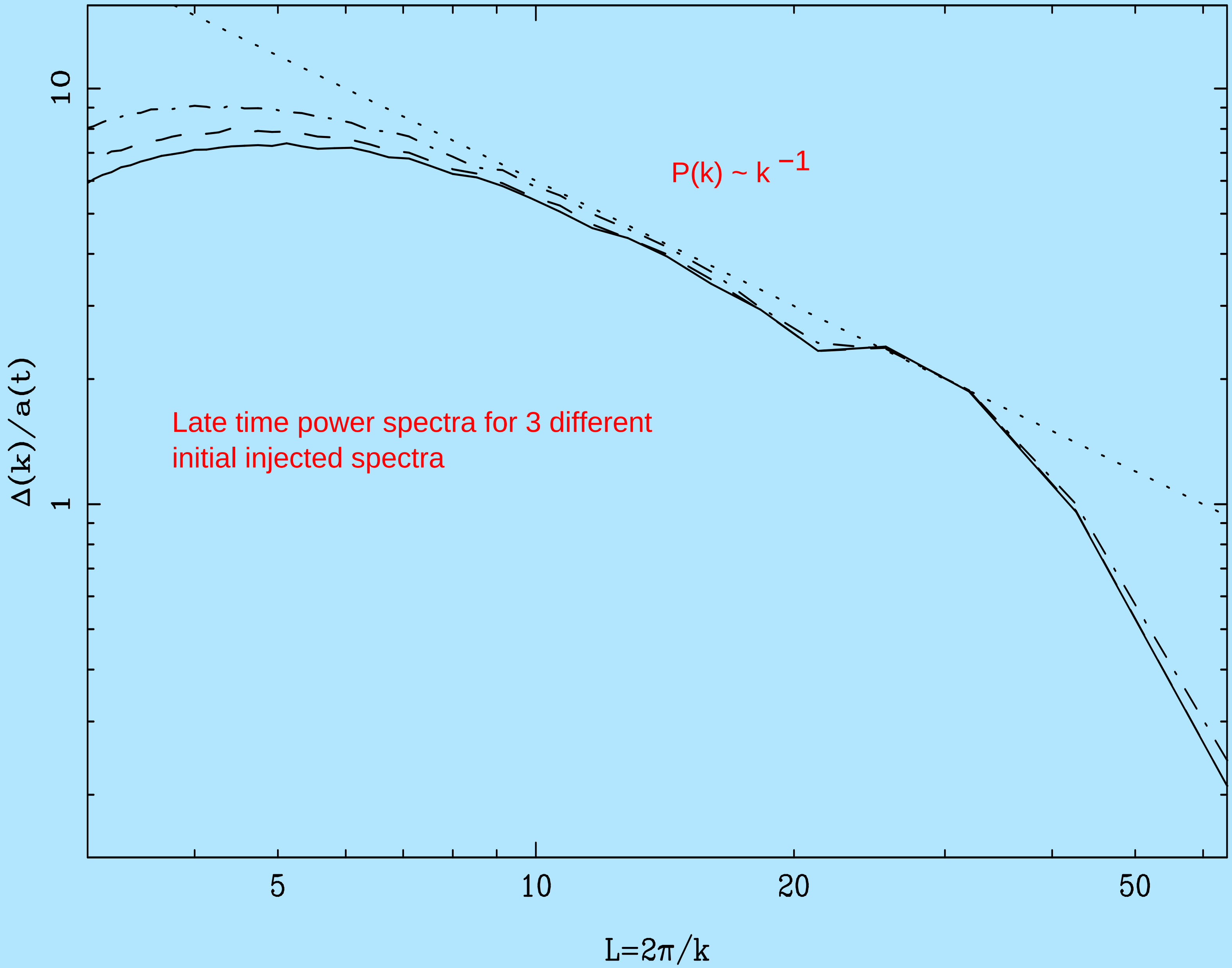

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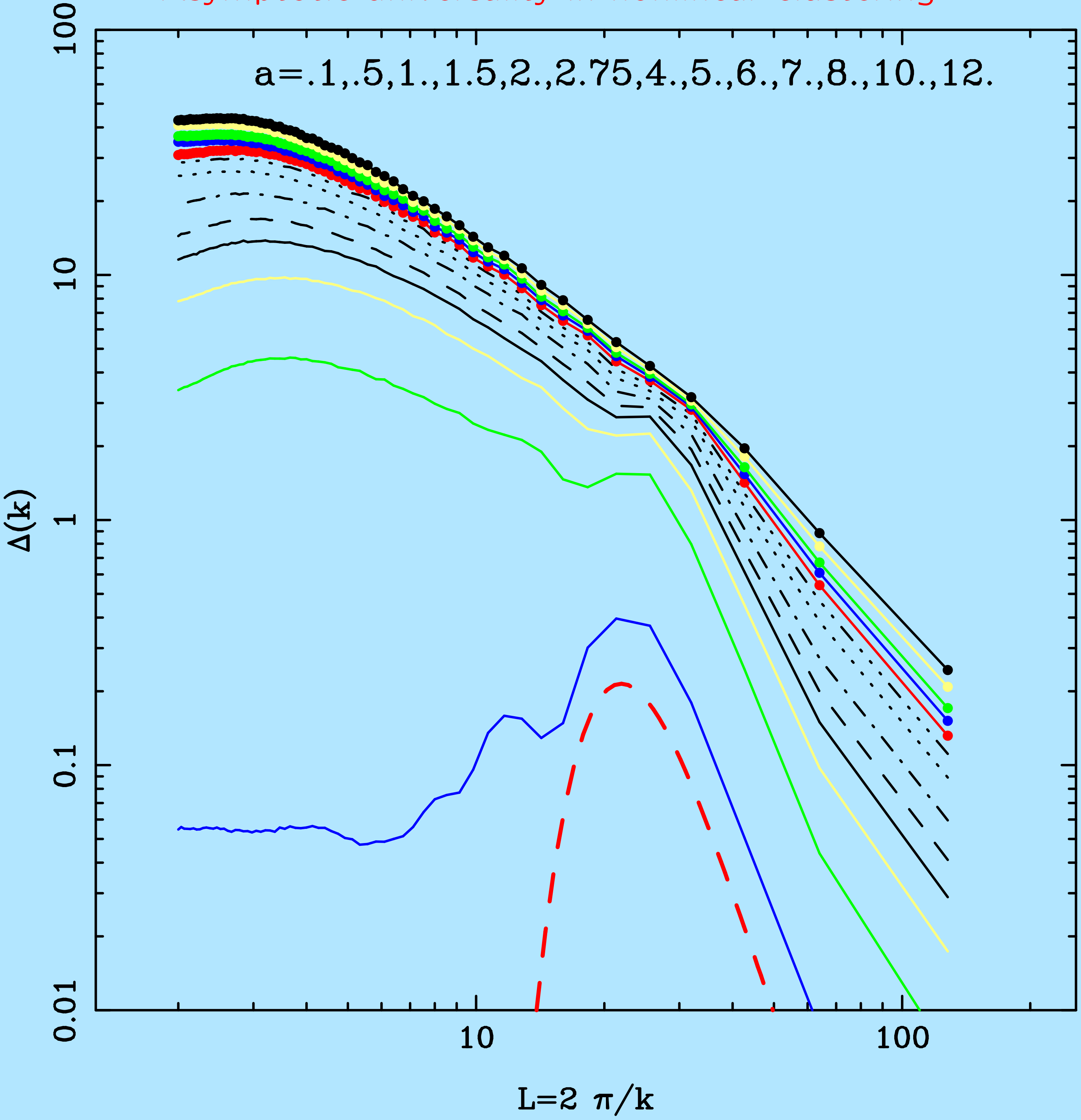
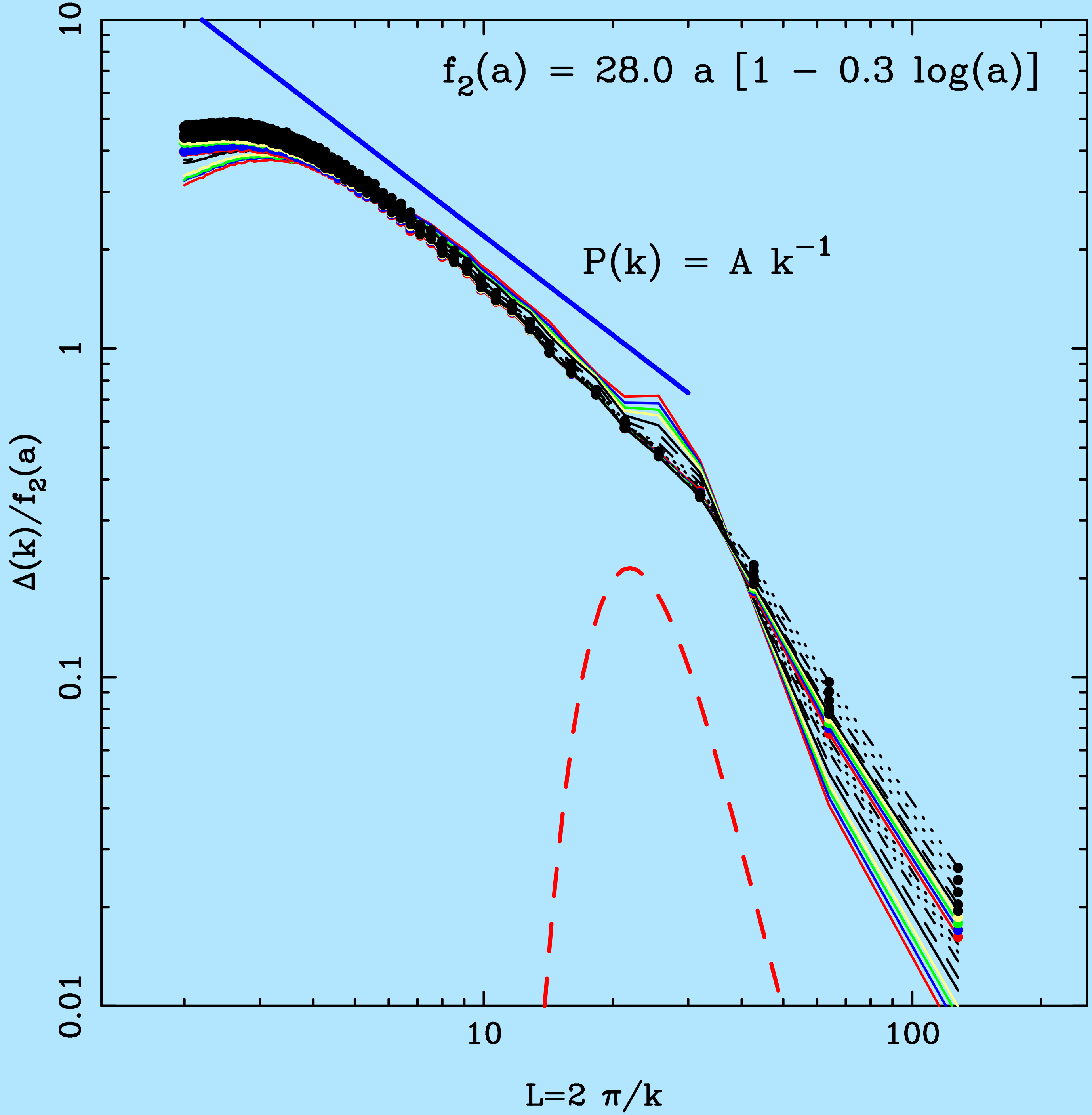
Asymptotic universality in nonlinear clustering

(T.P., S. Ray, 2006)





$f_2(a) = 28.0\ a\ [1 - 0.3\ \log(a)]$
$P(k) = A\ k^{-1}$
$\Delta(k)/f_2(a)$
$L=2\ \pi/k$
10
1
0.1
0.01
10
100

Summary

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- Chandra pioneered the use of statistical physics in the study of gravitating systems.
- His approach to this subject shows the characteristic rigour employed as a matter of policy rather than out of necessity.
- The subject is alive and well and still has several open questions especially in the context of cosmology.

Acknowledgements

- D. Lynden-Bell (IOA)
- D. Narasimha (TIFR)
- R. Nityananda (NCRA)
- S. Sridhar (RRI)
- K. Subramanian (IUCAA)