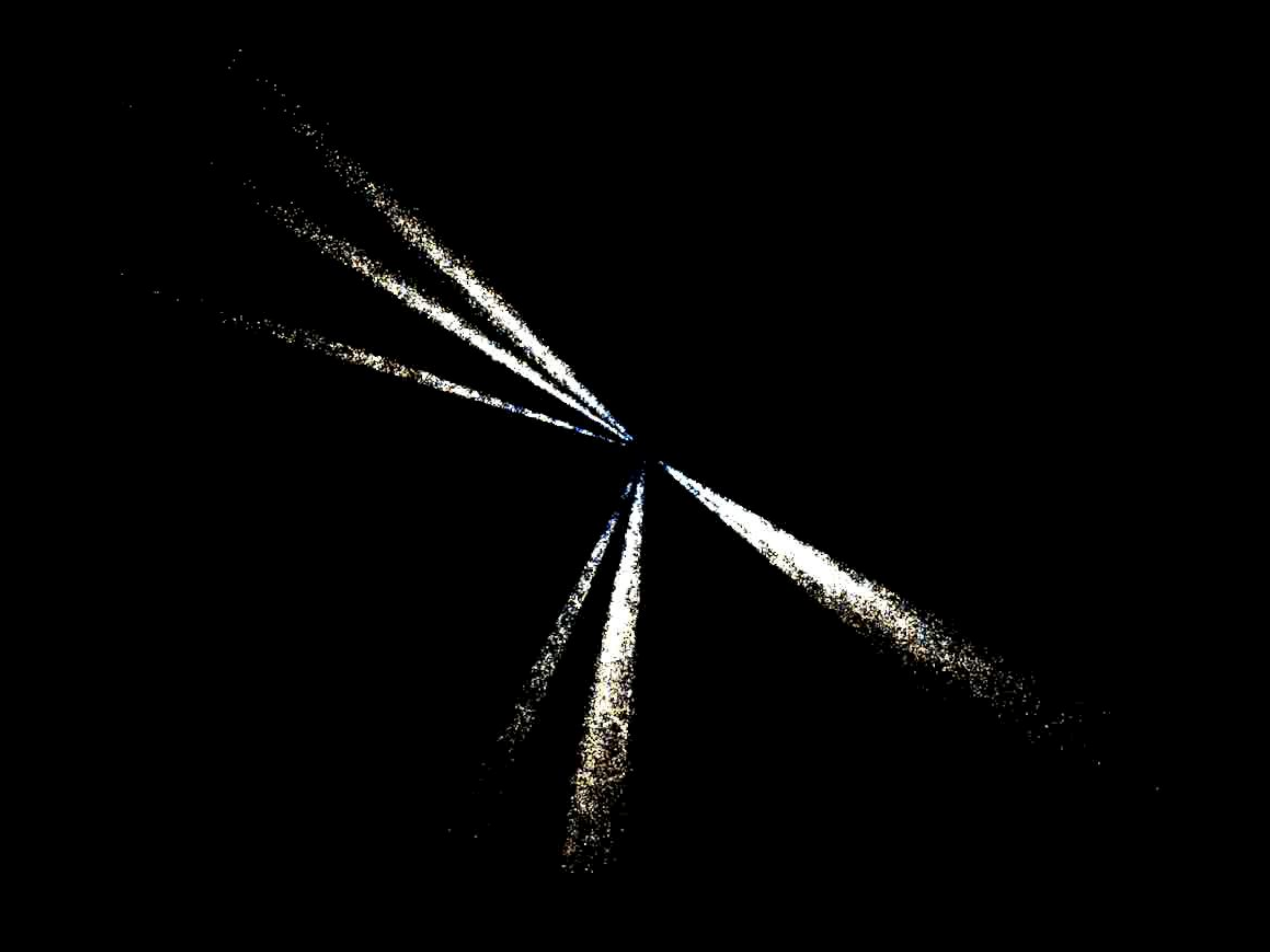
Symmetry and Stochasticity: Chandrasekhar and The Phenomenology of Large Scale Structure

Ravi K Sheth (Penn/ICTP)

- Motivation: A biased view of dark matters
  - Observations and simulations
- Gravitational Instability
  - The spherical (and tri-axial) collapse models
- The excursion set description
  - Percolation/branching process/coagulation descriptions
  - Halo abundances and clustering
  - The forest of merger trees
  - The nonlinear probability distribution function
  - Galaxy clustering

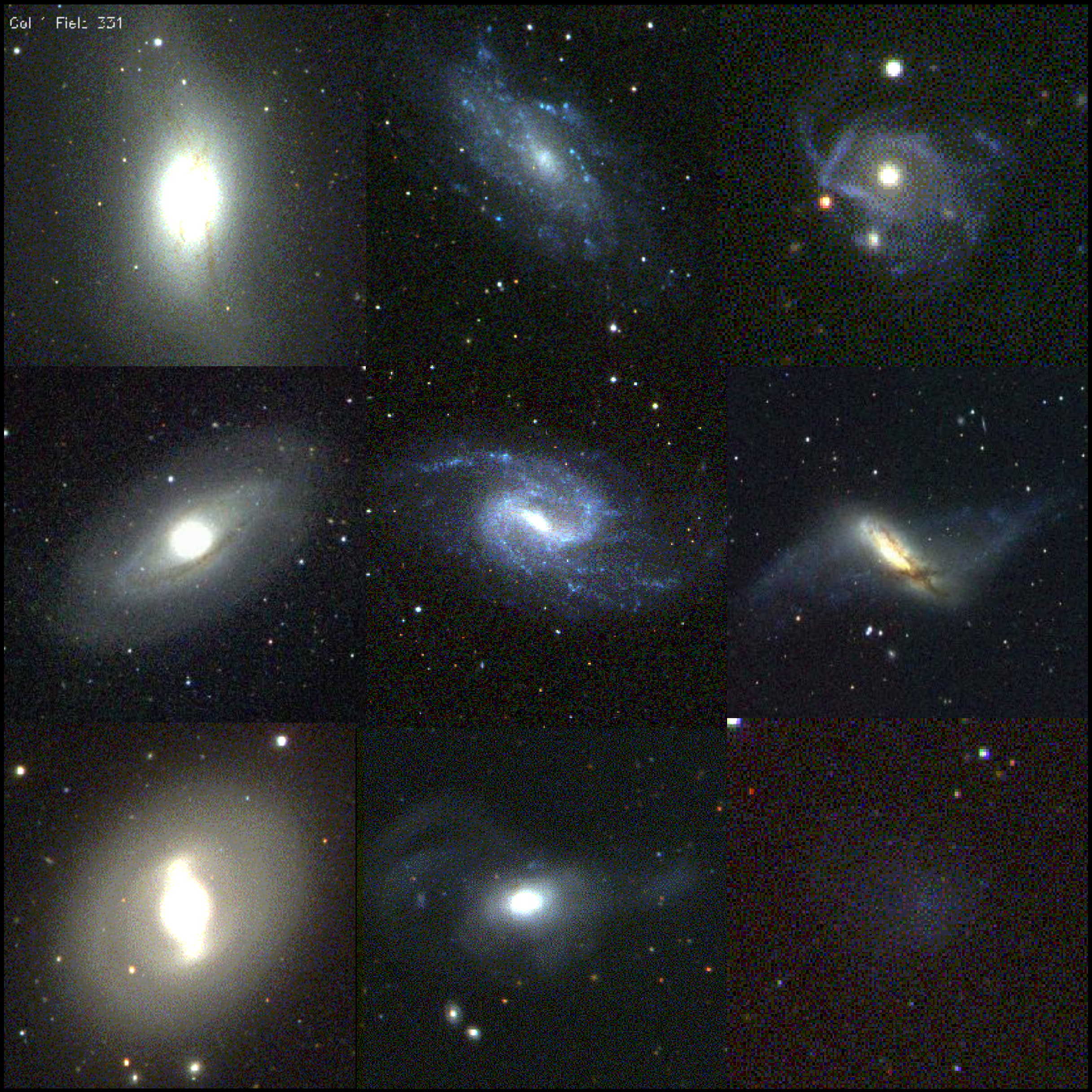


Galaxy Clustering varies with Galaxy Type

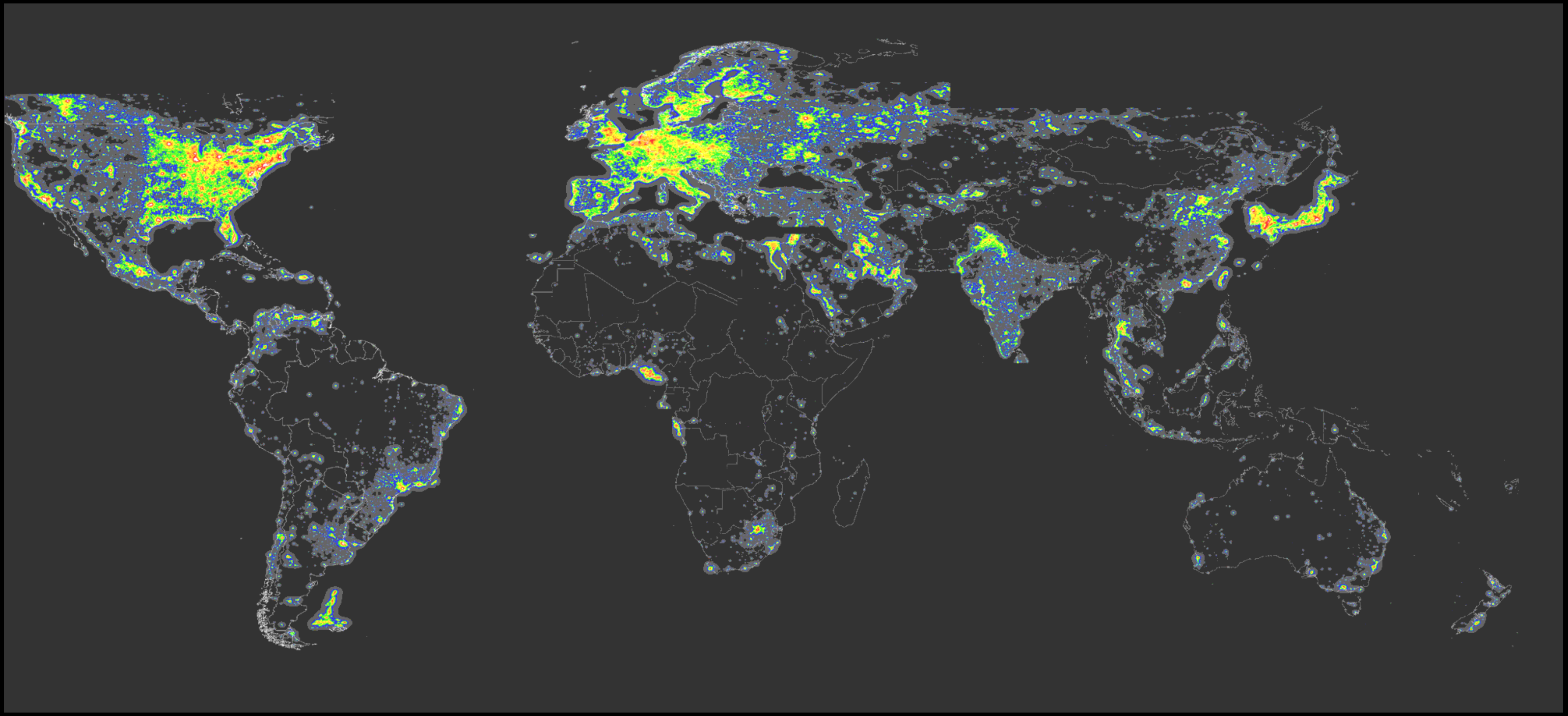
How is each galaxy population related to the underlying Mass distribution?

Bias depends upon Galaxy Color and Luminosity

Need large, carefully selected samples to study this (e.g. Norberg et al. 2002 2dFGRS; Zehavi et al. 2005)



Light is a biased tracer

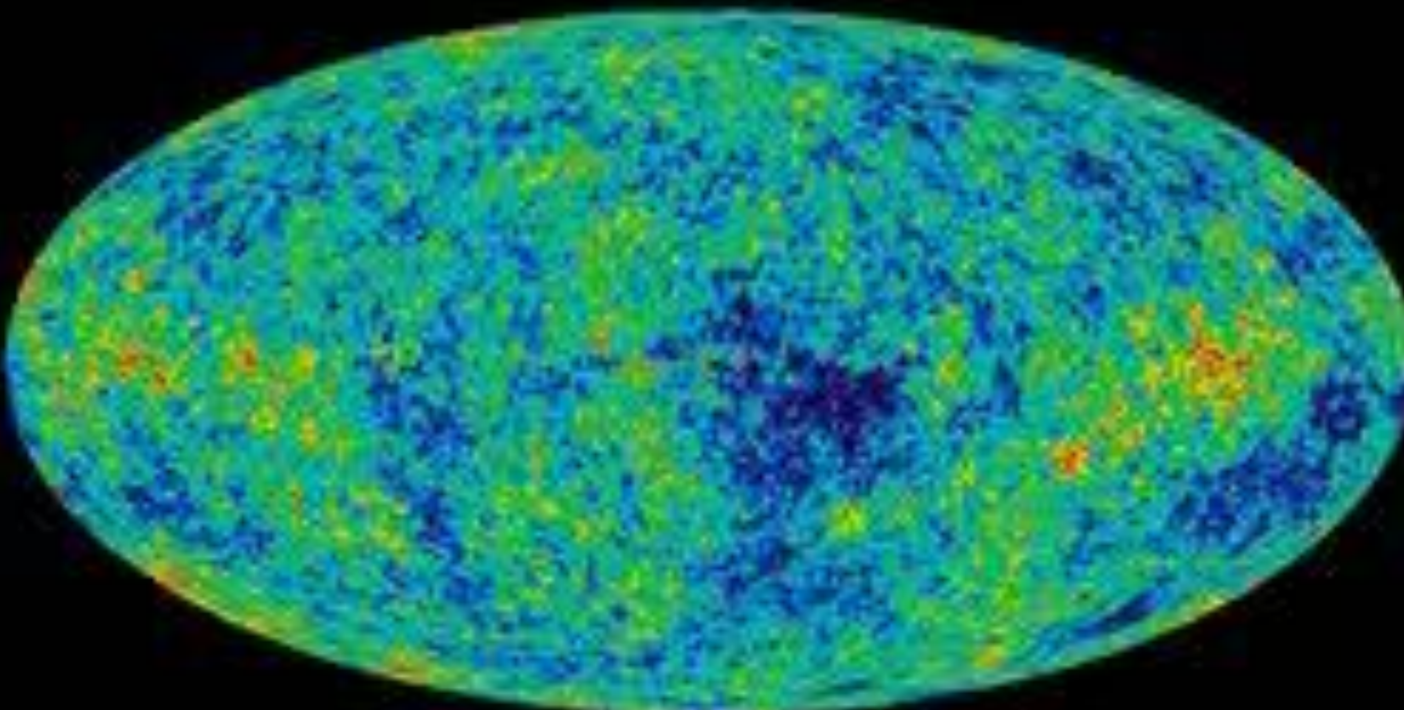


Understanding bias important for understanding mass

You can observe a lot
just by watching

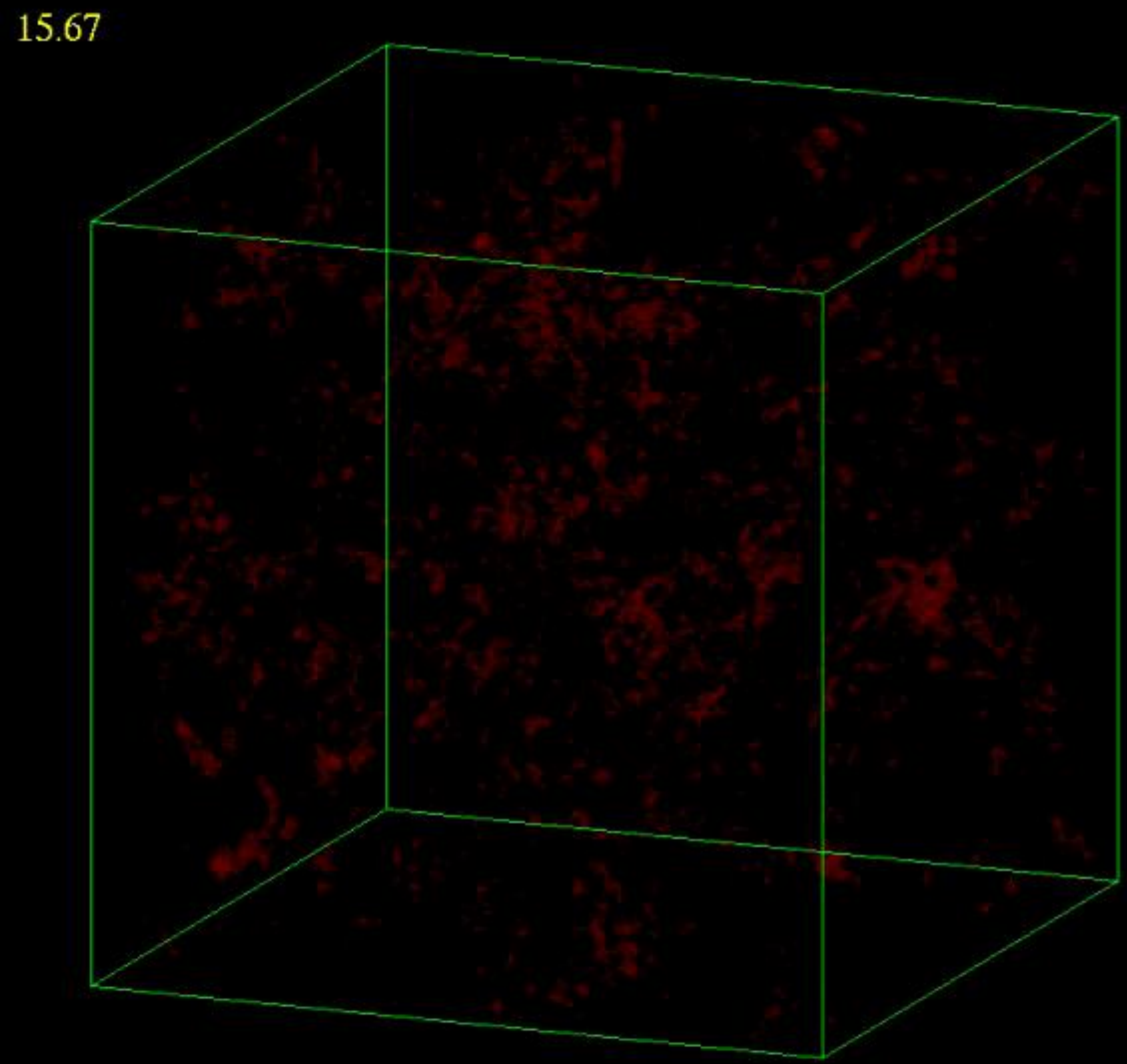
How to describe different point processes which are all built from the same underlying distribution?

THE HALO MODEL



Cold Dark Matter

- Simulations include gravity only (no gas)
- Late-time field retains memory of initial conditions
- Cosmic capitalism



Co-moving volume ~ 100 Mpc/h

It's a capitalist's life ...

- Most of the action is in the big cities
- Newcomers to the city are rapidly stripped of (almost!) all they have
- Encounters generally too high-speed to lead to long-lasting mergers
- Repeated 'harassment' can lead to change
- Real interactions take place in the outskirts
- A network exists to channel resources from the fields to feed the cities

Cold Dark Matter

- Cold: speeds are non-relativistic
- To illustrate, 1000 km/s $\times$10Gyr $\approx$ **10M**pc; from z~1000 to present, nothing (except photons!) travels more than ~ 10Mpc
- Dark: no idea (yet) when/where the stars light-up
- Matter: gravity the dominant interaction

N-body
simulations
of

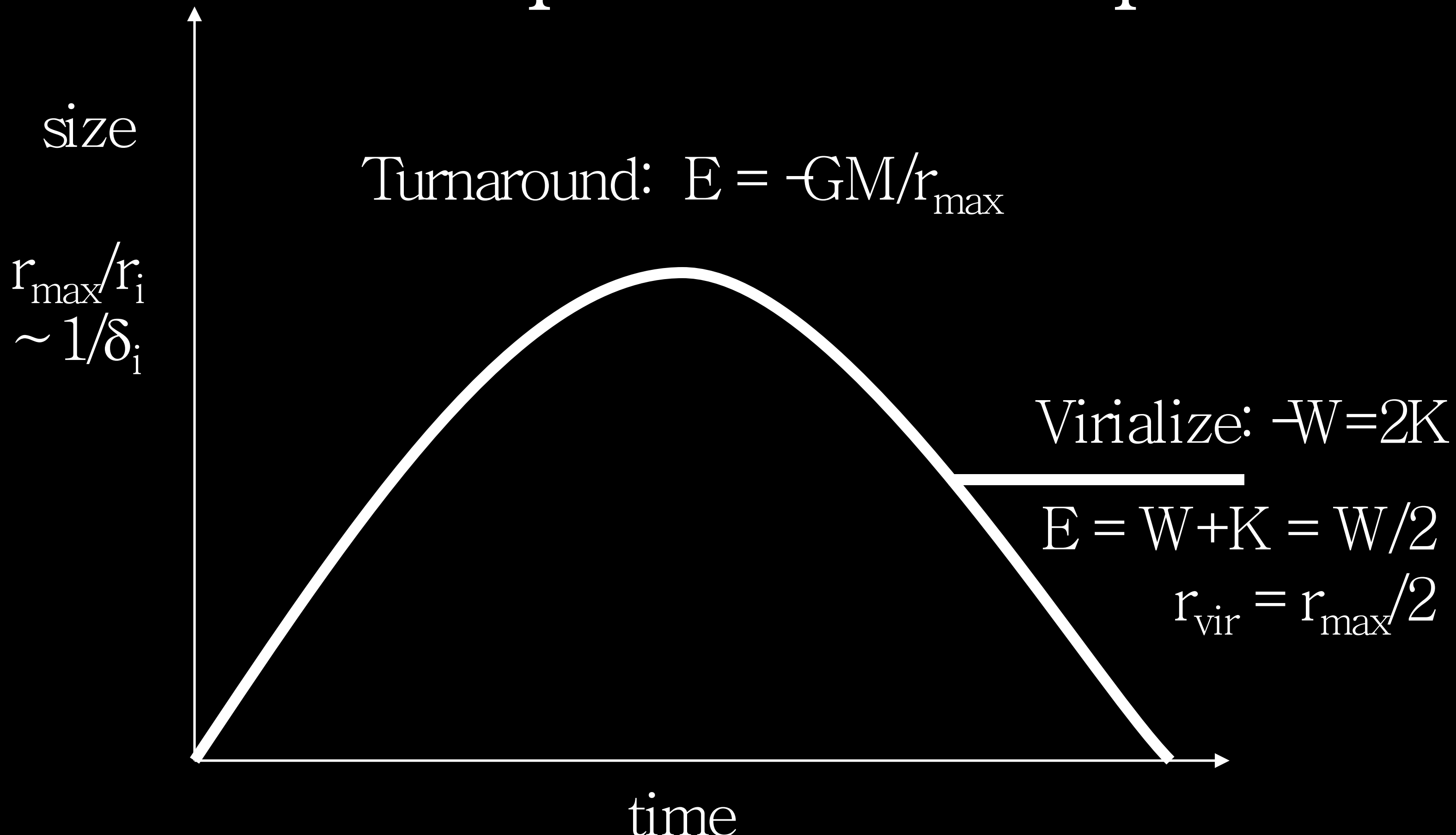
gravitational
clustering

in an
expanding
universe

Assume a spherical cow …

(Gunn & Gott 1972)

Spherical collapse



Modify gravity → modify collapse

At any given time,
nonlinear virialized objects:

Are the same density whatever their mass;
They formed from regions of similar initial overdensity, whatever their initial size

Exact Parametric Solution

(R_i/R) vs. θ and (t/t_i) vs. θ

very well approximated by…

$$(R_{\rm initial}/R)^3$$

$$= \text{Mass}/(\rho_{\rm com}\text{Volume})$$

$$= 1 + \delta \approx (1 - \delta_{\rm Linear}/\delta_{sc})^{-\delta sc}$$

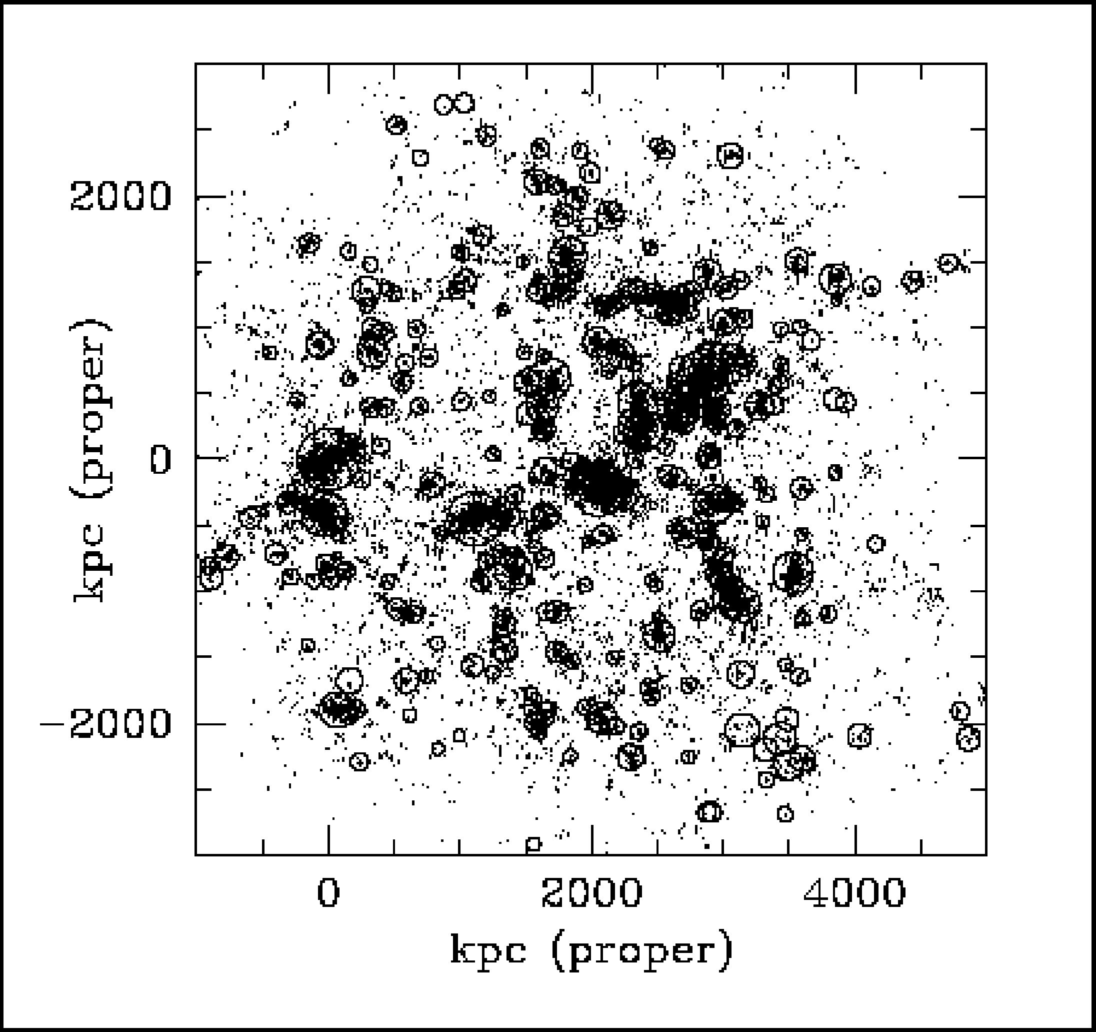
Dependence on cosmology from $\delta_{sc}(\Omega,\Lambda)$, but this is rather weak

$$1 + \delta \approx (1 - \delta_{Linear}/\delta_{sc})^{-\delta sc}$$

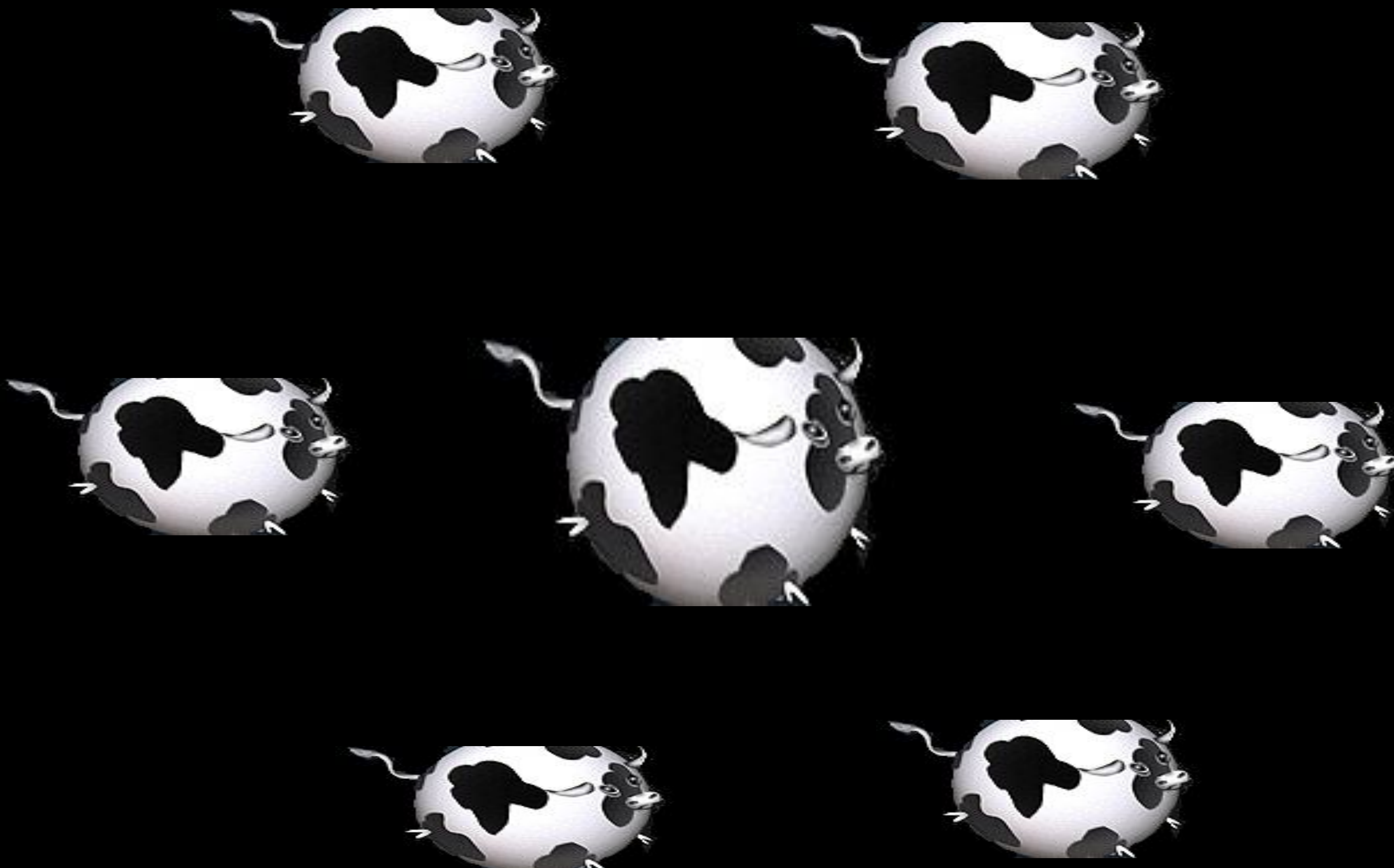
- As $\delta_{Linear} \to \delta_{sc}$, $\delta \to$ infinity
  - This is virialization limit
- As $\delta_{Linear} \to 0$, $\delta \approx \delta_{Linear}$
- If $\delta_{Linear} = 0$ then $\delta = 0$
  - This does not happen in modified gravity models where $D(t) \to D(k,t)$!
  - Related to loss of Birkhoff's theorem when r^{-2} lost?

Spherical evolution model

- 'Collapse' depends on initial over-density Δ_i; same for all initial sizes
- Critical density depends on cosmology
- Final objects all have same density, whatever their initial sizes
- Collapsed objects called halos; ~200× denser than background, whatever their mass

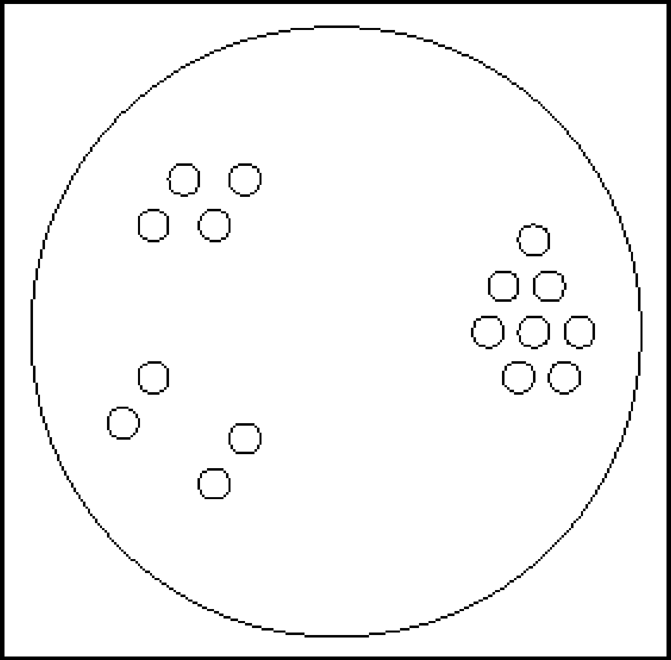


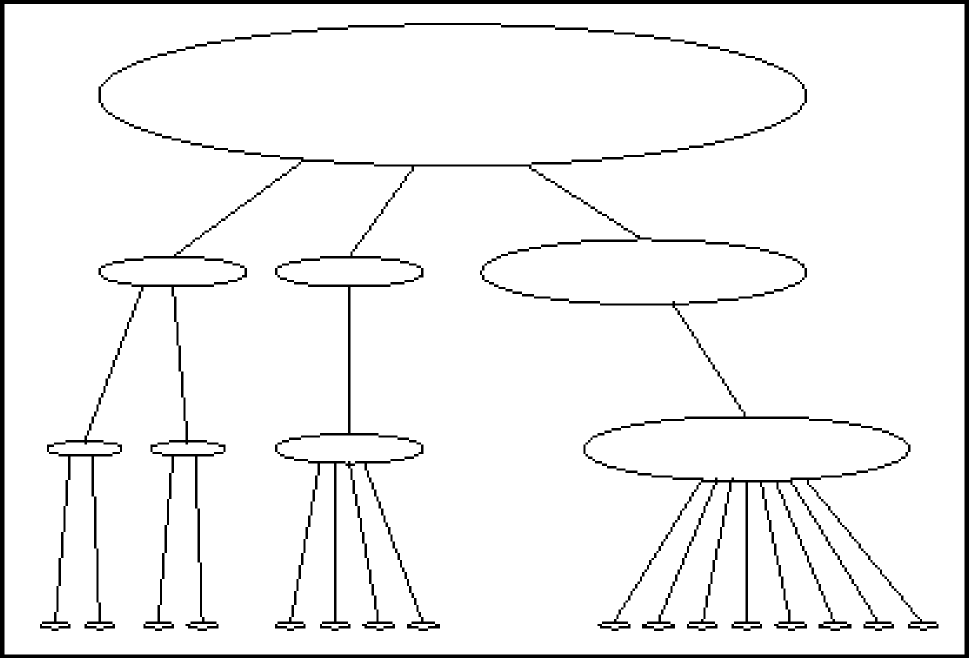
(Figure shows particles at z~2 which, at z~0, are in a cluster)



Assume a spherical herd of spherical cows…

Initial spatial distribution within patch (at z~1000)...





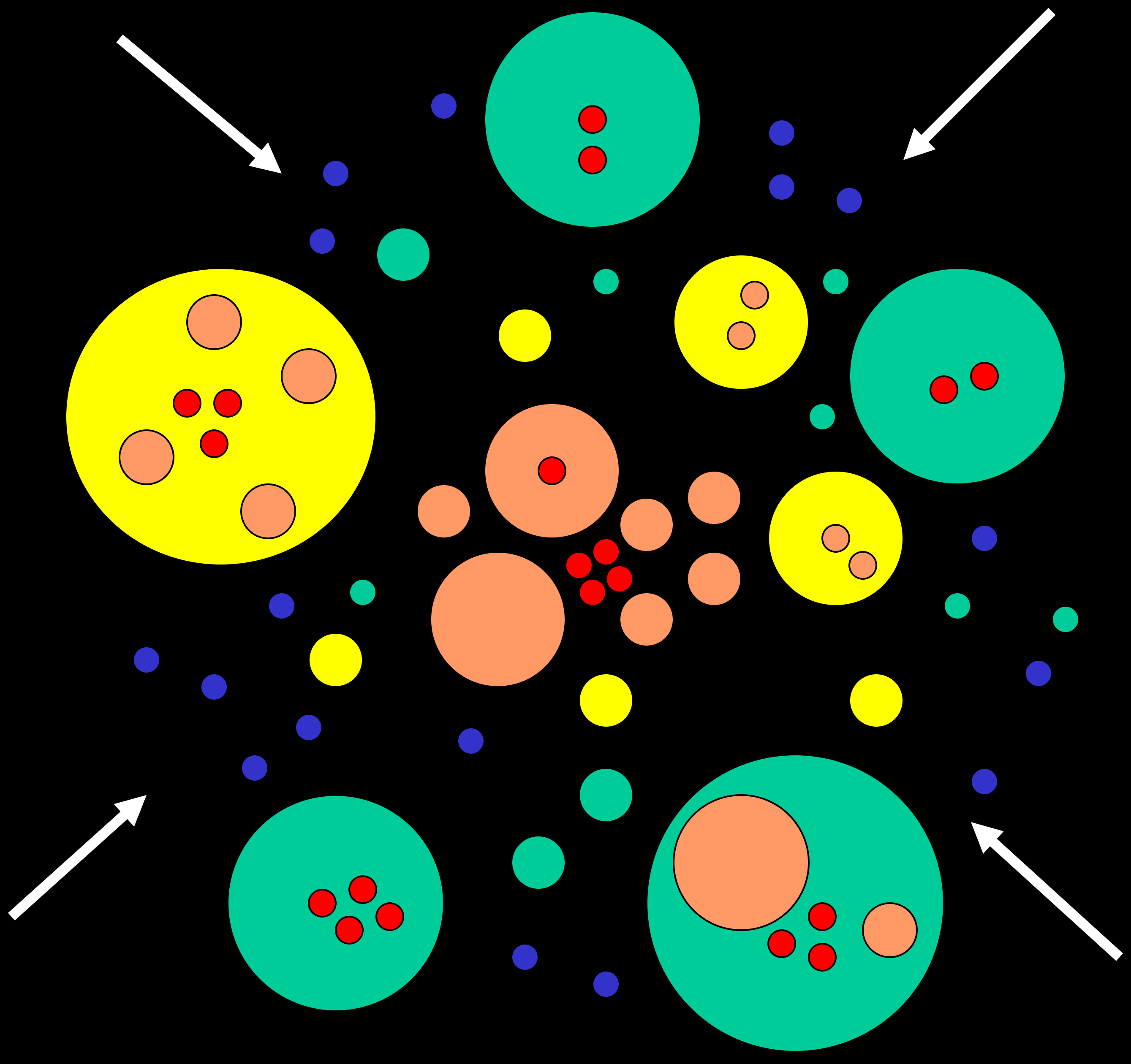
…stochastic (initial conditions Gaussian random field); study 'forest' of merger history 'trees'.

…encodes information about subsequent 'merger history' of object

(Mo & White 1996; Sheth 1996)

Schematic view of merger history of central object

To this, add dynamical friction, tidal stripping, interactions, etc.



Motivation for models …

percolation/branching process
coagulation/fragmentation
excursion set/random walk
(Smoluchowski + Chandrasekhar)

… which all give the same answer

Goal:

Use initial conditions (CMB)

+

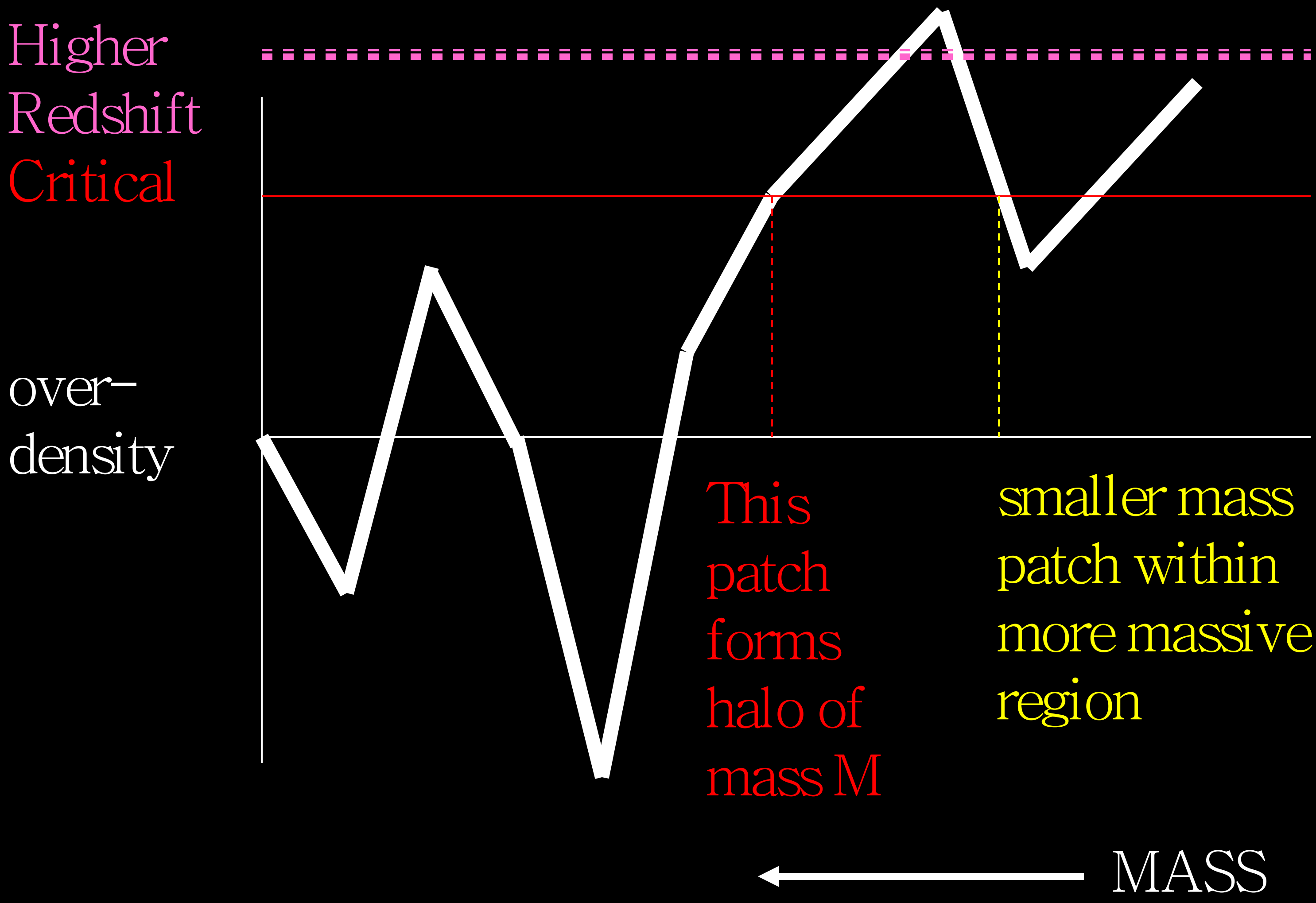
model of nonlinear gravitational clustering

to make inferences about late-time, nonlinear structures

THE EXCURSION SET APPROACH

(Epstein 1983; Bond et al. 1991; Lacey & Cole 1993;
Sheth 1998; Sheth & van de Weygaert 2004; Shen et al. 2006)

The Random Walk Model



From Walks to Halos: Ansätze

- $f(\delta_c, s)ds$ = fraction of walks which first cross $\delta_c(z)$ at s

  $\approx$ fraction of initial volume in patches of comoving volume $V(s)$ which were just dense enough to collapse at z

  $\approx$ fraction of initial mass in regions which each initially contained $m = \rho V(1+\delta_c) \approx \rho V(s)$ and which were just dense enough to collapse at z (ρ is comoving density of background)

  $\approx dm\, m\, n(m,\delta_c)/\rho$

Random walk with absorbing barrier

- f(first cross δ_1 at s) = ${}_0\int^s dS\, f$(first cross δ_0 at S)
  $\times f$(first cross δ_1 at s | first cross δ_0 at S)
  (where $\delta_1 > \delta_0$ and $s > S$)
- But second term is function of $\delta_1 - \delta_0$ and $s - S$
  - because subsequent steps independent of previous ones, so statistics of subsequent steps are simply a shift of origin – a key assumption we will return to later

- $$f(\delta_1,s) = {}_0\int^s dS\, f(\delta_0,S)\, f(\delta_1-\delta_0|s-S)$$

- To solve ...

First-crossing distributions

- 29 March 1900, Louis Bachelier defends PhD thesis and mathematical finance is born (crossing of constant barrier ~ pricing of options and derivatives)

- Schrödinger studied first crossings of linear barrier

- … take Laplace Transform of both sides:
- $\mathcal{L}(\delta_1,t) = {}_0\int^{\infty} ds\, f(\delta_1,s) \exp(-ts)$

  $= {}_0\int^{\infty} ds \exp(-ts)\ {}_0\int^{s} dS\, f(\delta_0,S)\, f(\delta_1-\delta_0,s-S)$

  $= {}_0\int^{\infty} dS\, f(\delta_0,S)\, e^{-tS}\ {}_{s-S}\int^{\infty} ds\, f(\delta_1-\delta_0,s-S)\, e^{-t(s-S)}$

  $= \mathcal{L}(\delta_0,t)\ \mathcal{L}(\delta_1-\delta_0,t)$
- Solution must have form: $\mathcal{L}(\delta_1,t) = \exp(-C\delta_1)$
- After some algebra: $\mathcal{L}(\delta_1,t) = \exp(-\delta_1\sqrt{2t})$
- Inverting this transform yields:
- $f(\delta_1,s)\, ds = (\delta_1^2/2\pi s)^{\frac{1}{2}}\ \exp(-\delta_1^2/2s)\, ds/s$
- Notice: few walks cross before $\delta_1^2=2s$

Who needs Laplace transforms?

- $p(\delta_1, s) = {}_0\int^S dS\, f(\delta_0, S)\;\; p(\delta_1, s\,|\,\text{first cross } \delta_0 \text{ at } S)$
- (where $\delta_1 > \delta_0$ and $s > S$)
- But second term in integral is just $p(\delta_1 - \delta_0 \mid s - S)$
  - subsequent steps independent of previous ones, so statistics of subsequent steps are simply a shift of origin
- So

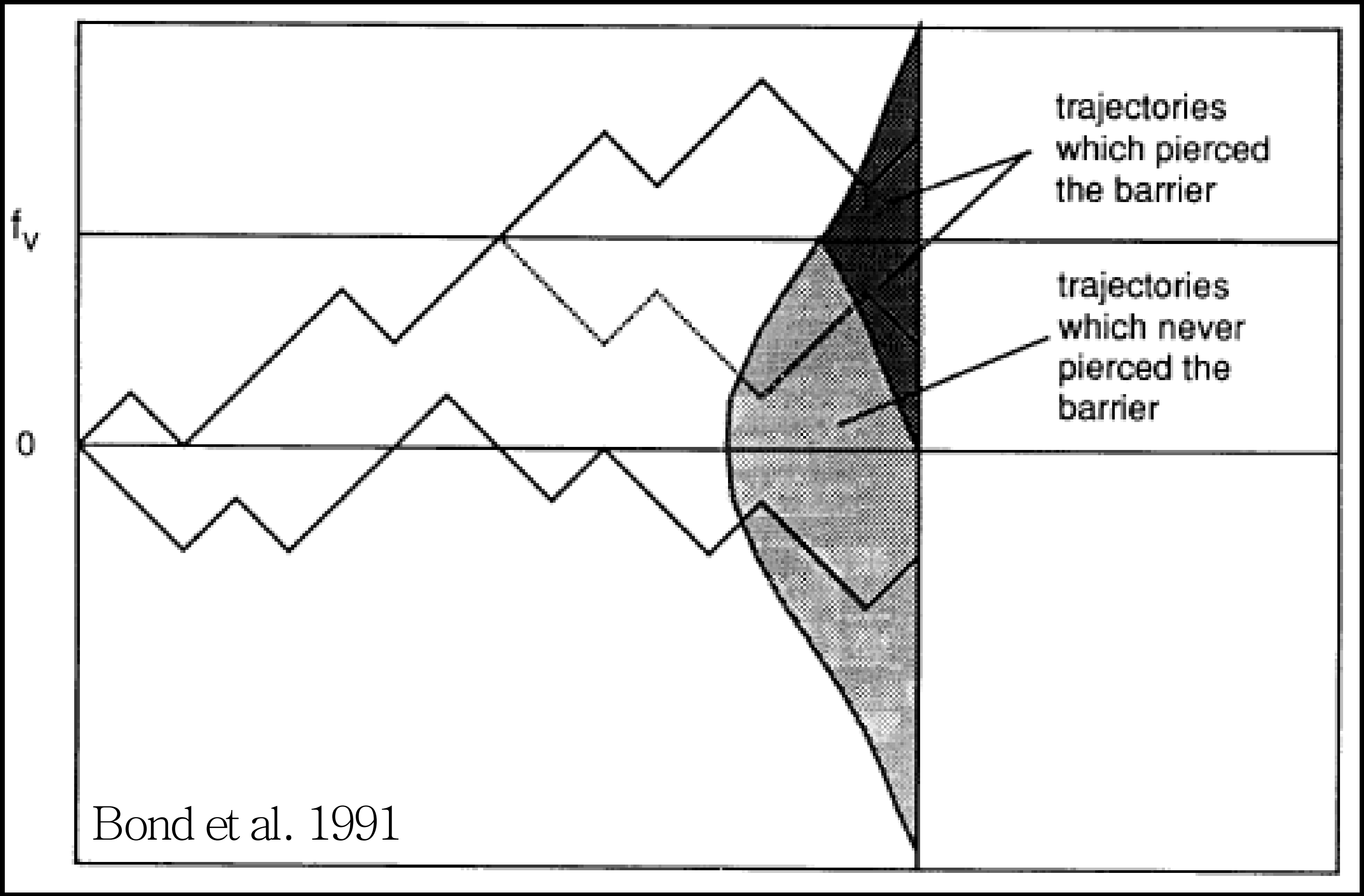
$$ {}_{\delta_0}\int^{\infty} d\delta_1\;\; p(\delta_1\,,\, s) = {}_0\int^S dS\, f(\delta_0, S)\; {}_{\delta_0}\int^{\infty} d\delta_1\, p(\delta_1 - \delta_0 \mid s - S) $$

$$ = {}_0\int^S dS\, f(\delta_0, S)\;\times\; \tfrac{1}{2} $$

- And so

$$ \mathrm{d}\,\mathrm{erfc}(\sqrt{\delta_1^2/2s})/\mathrm{d}s = f(\delta_0, s) \times \tfrac{1}{2} $$

Chandra's factor of 2



The Mass Function

- $f(\delta_c, s)\, ds = (\delta_c^2/2\pi s)^{\frac{1}{2}}\ \exp(-\delta_c^2/2s)\ ds/s$

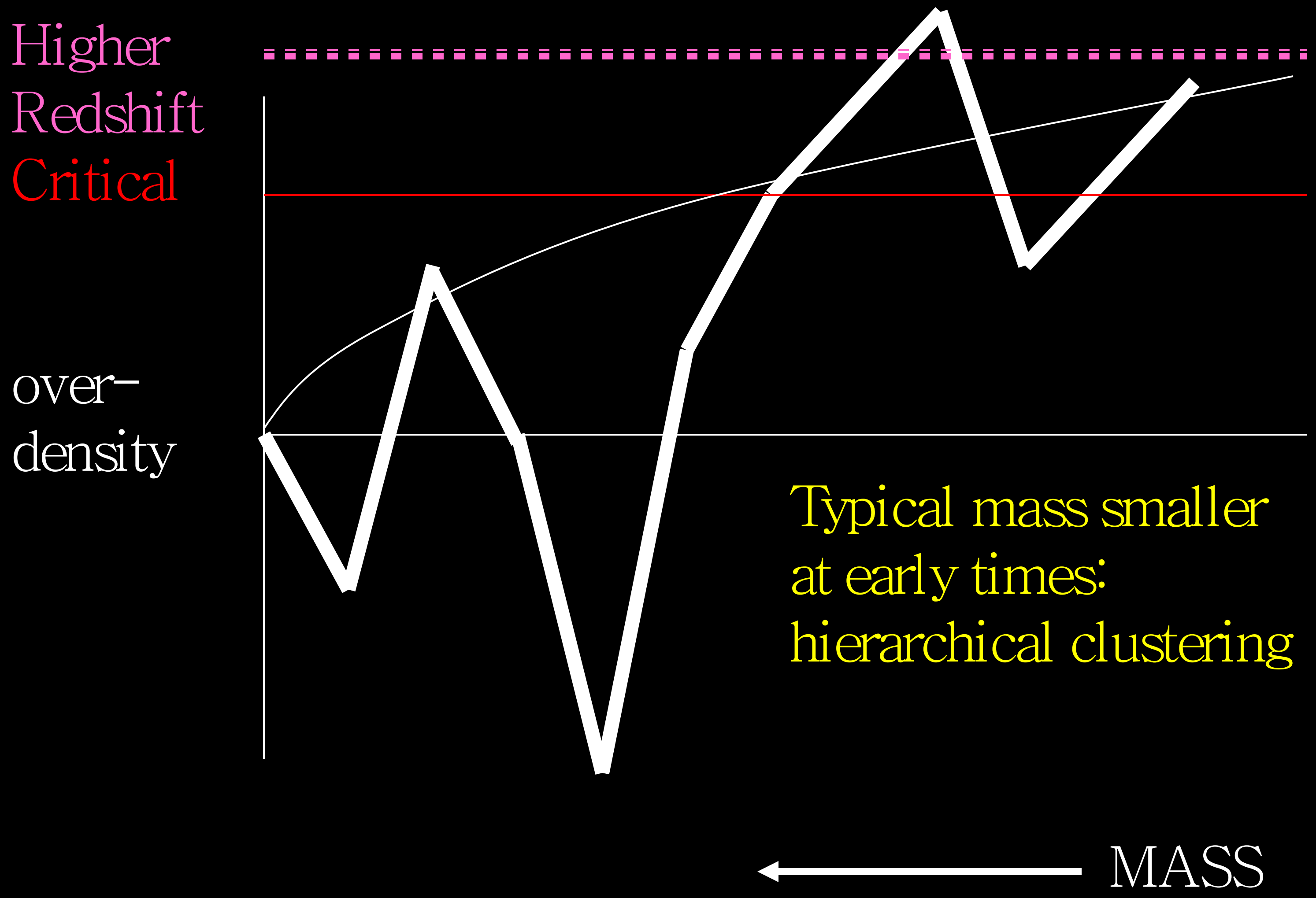
  CHANDRASEKHAR 1943

- For power-law $P(k)$: $\delta_c^2/s = (M/M_*)^{(n+3)/3}$

- $n(m,\delta_c)\, dm = (\rho/m)/\sqrt{2\pi}\ (n+3)/3\ dm/m\ (M/M_*)^{(n+3)/6} \exp[-(M/M_*)^{(n+3)/3}/2]$

(Press & Schechter 1974; Bond et al. 1991)

The Random Walk Model

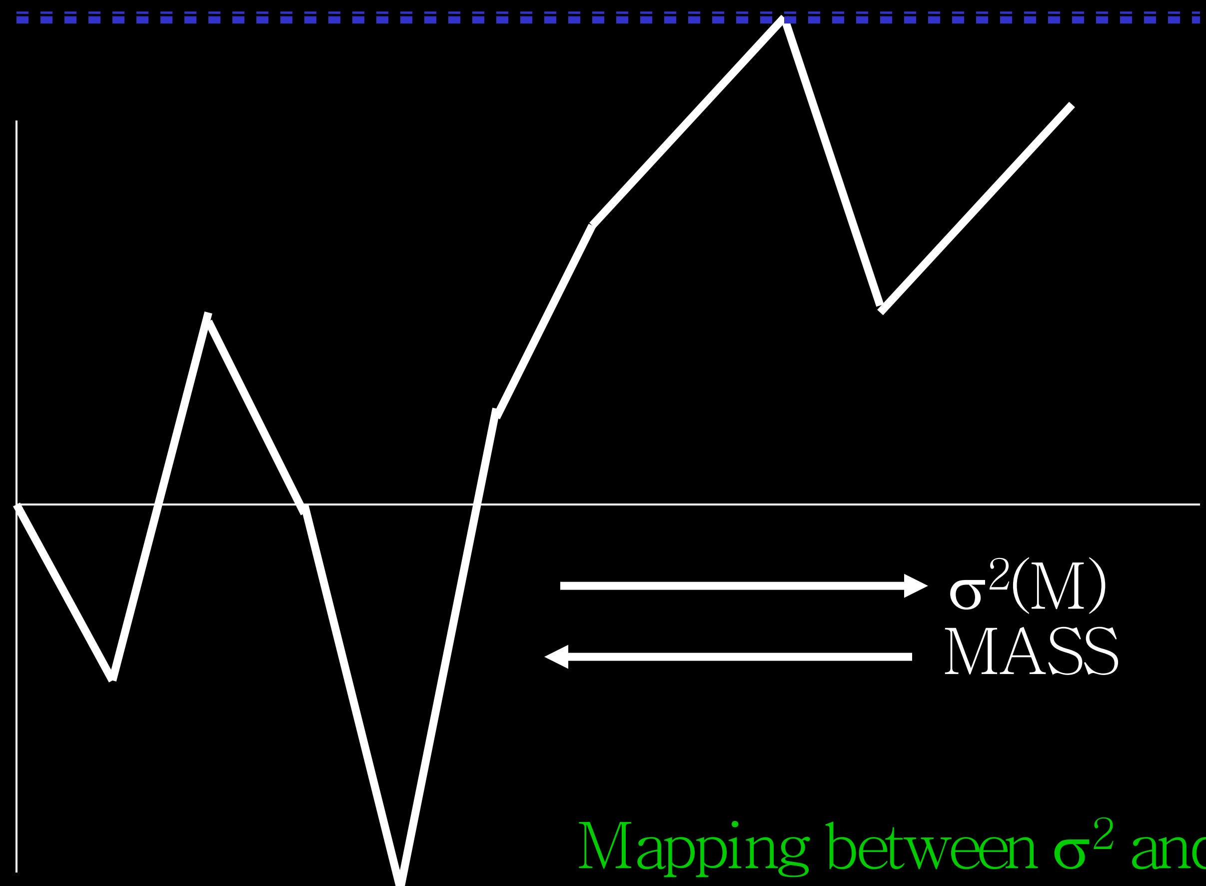


Excursion Set Approach

High-z

Low-z

over-density



Time evolution of barrier depends on cosmology

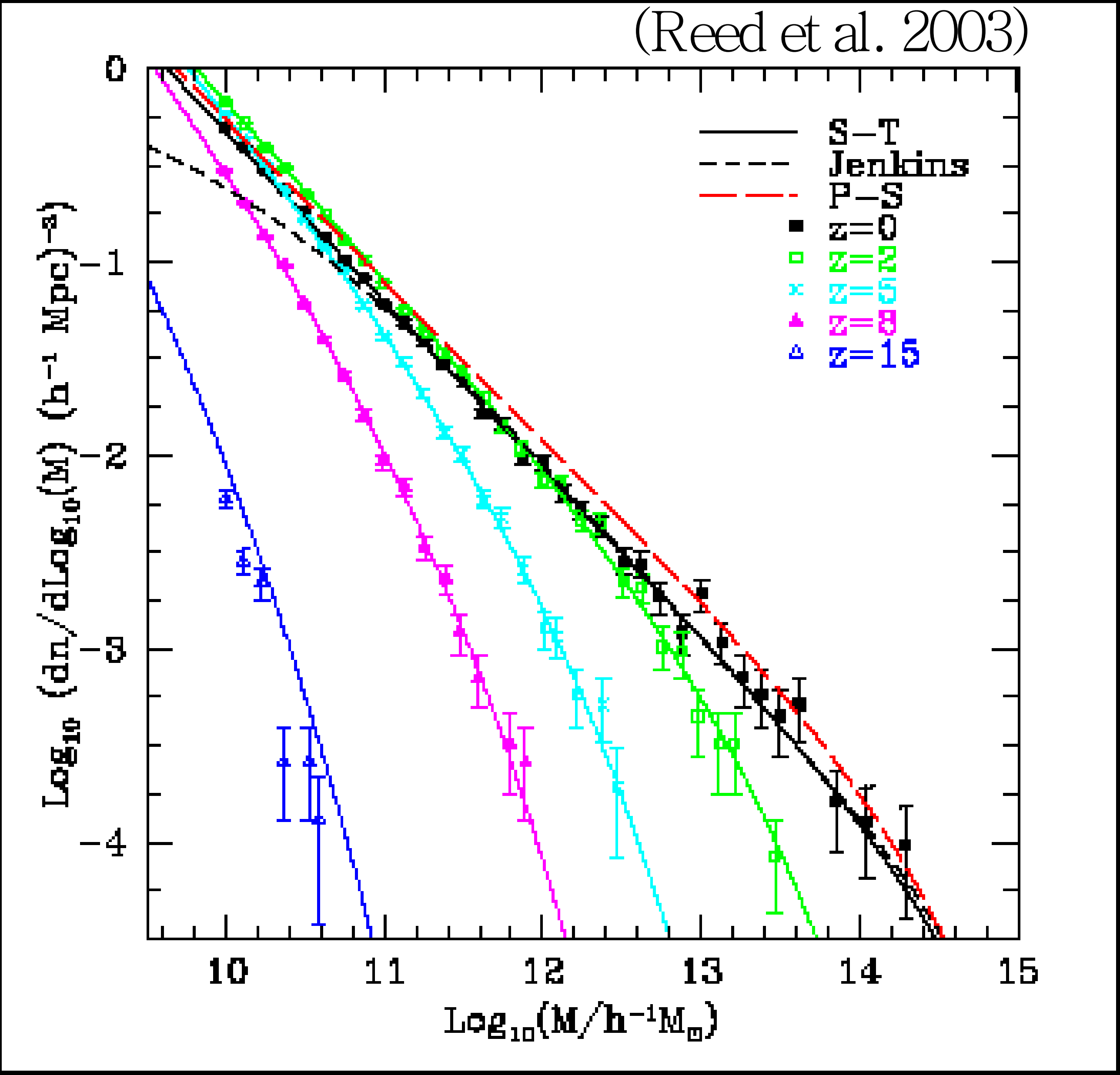
Mapping between σ^2 and M depends on *P(k)*

Simplification because…

- Everything local
- Evolution determined by cosmology (competition between gravity and expansion)
- Statistics determined by initial fluctuation field: since Gaussian, statistics specified by initial power-spectrum *P(k)*
- Fact that only very fat cows are spherical is a detail (*crucial* for precision cosmology); in excursion set approach, mass-dependent barrier height increases with distance along walk

The Halo Mass Function

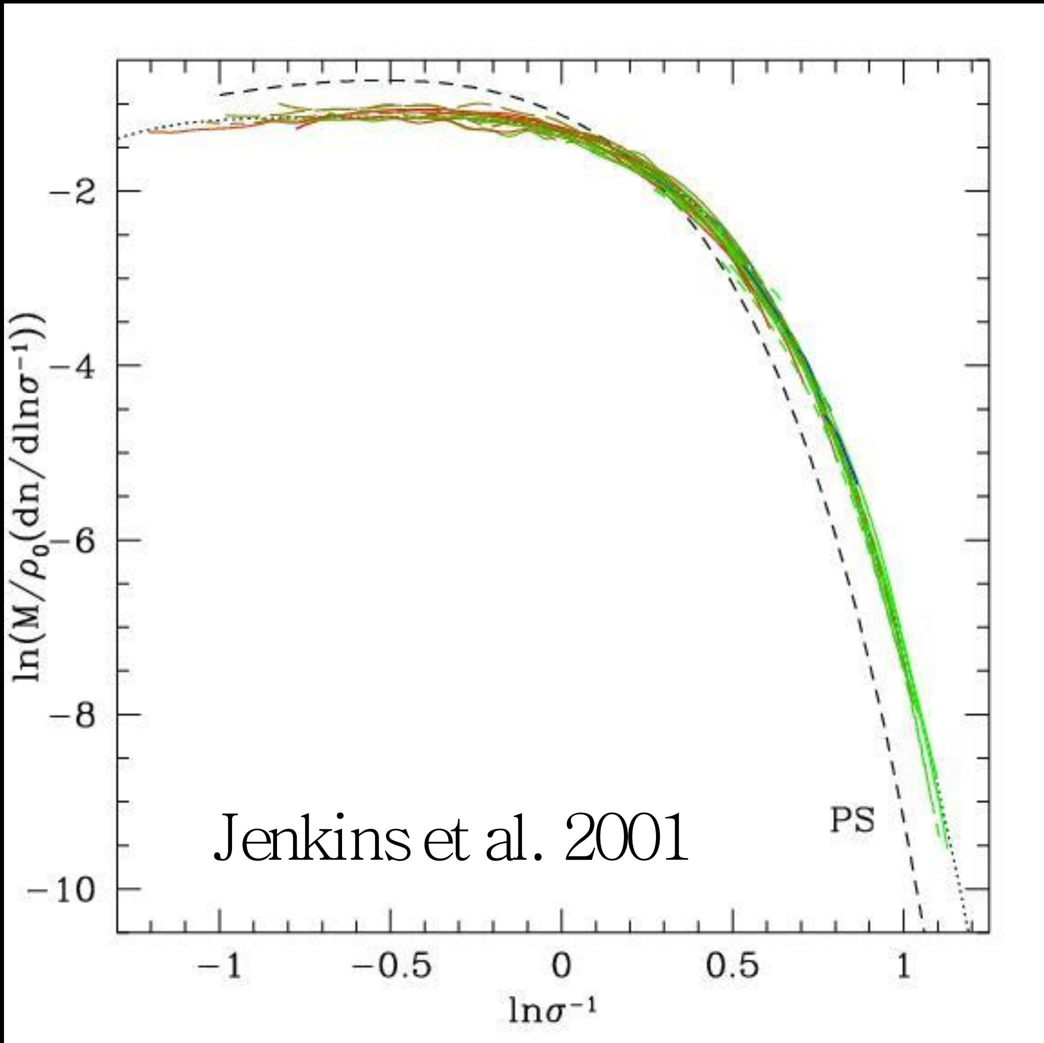
- Small halos collapse/virialize first
- Can also model halo spatial distribution
- Massive halos more strongly clustered



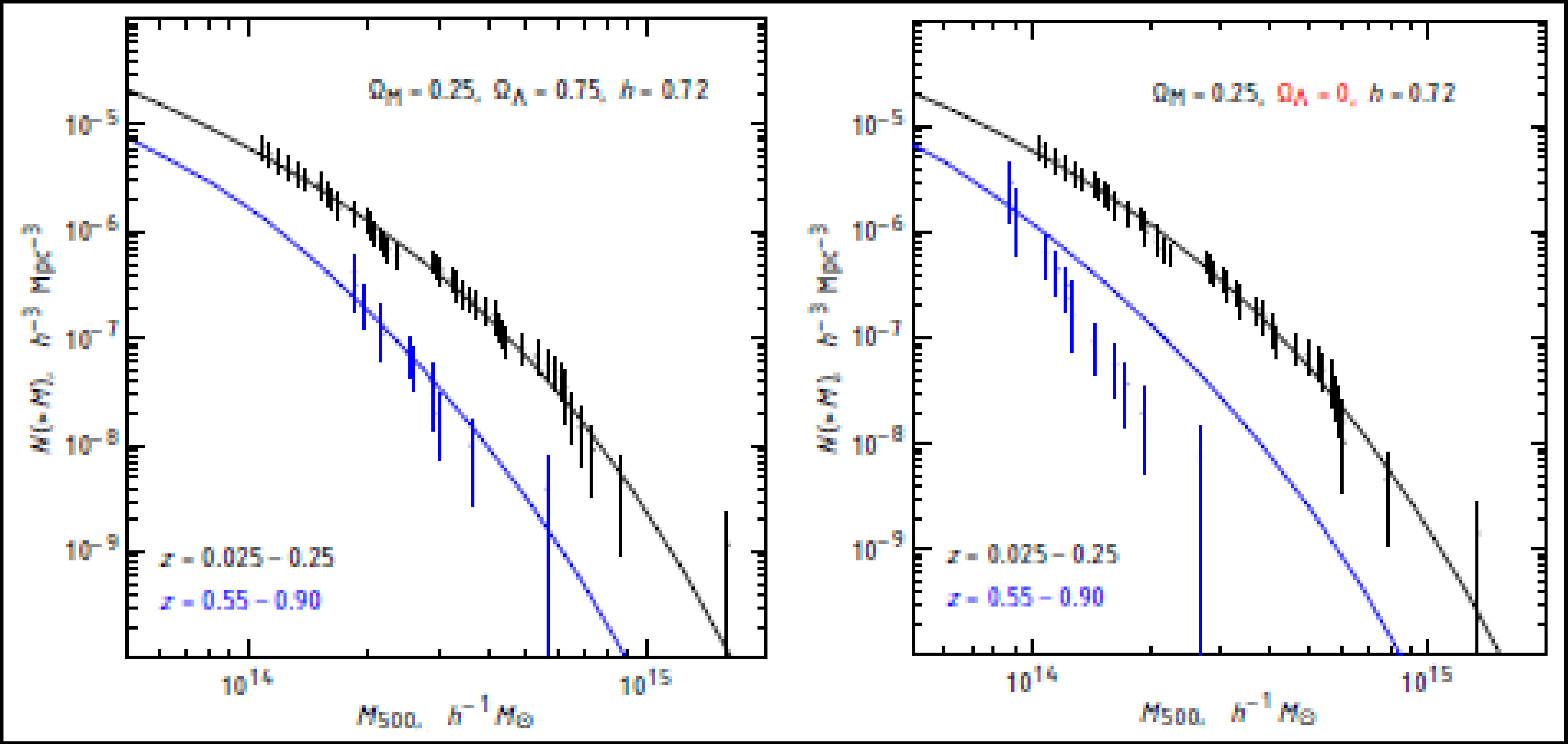
(current parametrizations by Sheth & Tormen 1999; Jenkins etal. 2001)

Universal form?

- Spherical evolution (Press & Schechter 1974; Bond et al. 1991)
- Ellipsoidal evolution (Sheth & Tormen 1999; Sheth, Mo & Tormen 2001)
- Simplifies analysis of cluster abundances (e.g. Xray, SZ, Opt)



X-ray cluster cosmology

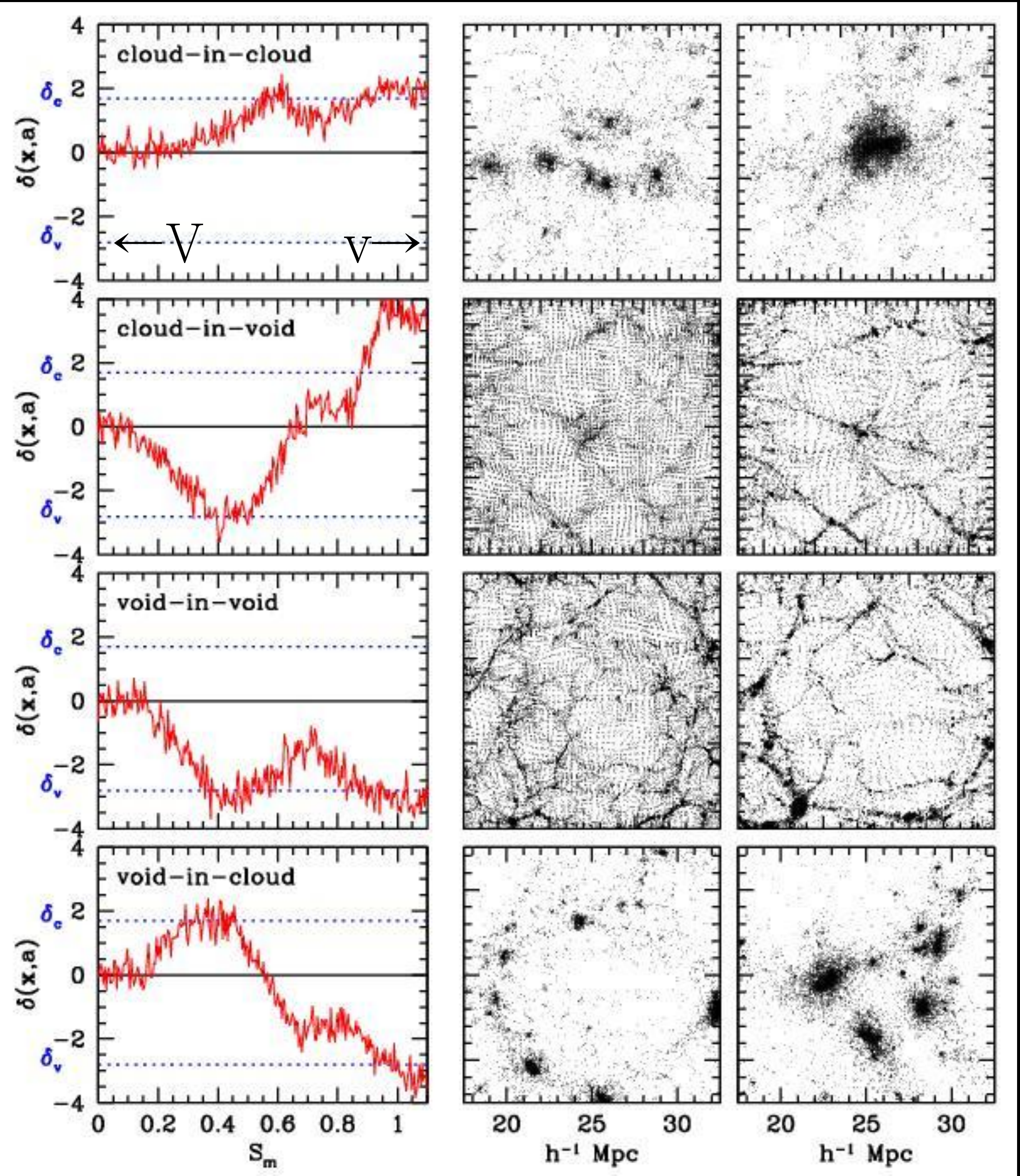


Vikhlinin et al. 2009

Random Walks

- Gaussian initial fluctuation field + spherical evolution model = hierarchical growth of structure

(Bond et al. 1991; Lacey & Cole 1993; Sheth 1998; Sheth & van de Weygaert 2004)



Voids: Much ado about nothing

- To account for both void-in-void and void-in-cloud problems, require two barriers:
- Of walks which first cross δ_v at s, remove those which first crossed δ_c

$$F_v(s) = f_v(s) - {}_0\int^s dS\, F_c(S)\, f_v(s|S)$$

$$F_c(s) = f_c(s) - {}_0\int^s dS\, F_v(S)\, f_c(s|S)$$

- Again, it is the Laplace transforms which behave intuitively, and allow solution

Inverting Laplace transform:

$$S\mathcal{F}(S, \delta_v, \delta_c) = \sum_{j=1}^{\infty} \frac{j^2\pi^2\mathcal{D}^2}{\delta_v^2/S} \frac{\sin(j\pi\mathcal{D})}{j\pi} \exp\left(-\frac{j^2\pi^2\mathcal{D}^2}{2\delta_v^2/S}\right)$$

Who needs Laplace transforms?

Chandrasekhar's reflection principle:

$$F_v(s) = f_v(s) - f_{c+t}(s) + f_{v+2t}(s) - \ldots$$

$$= \text{zig} - \text{zigzag} + \text{zigzagzig} - \ldots$$

(where $\delta_t = \delta_c + |\delta_v|$)

THE NONLINEAR PROBABILITY DISTRIBUTION FUNCTION OF DARK MATTER

(Sheth 1998; Lam & Sheth 2008)

Recall: Spherical evolution very well approximated by 'deterministic' mapping …

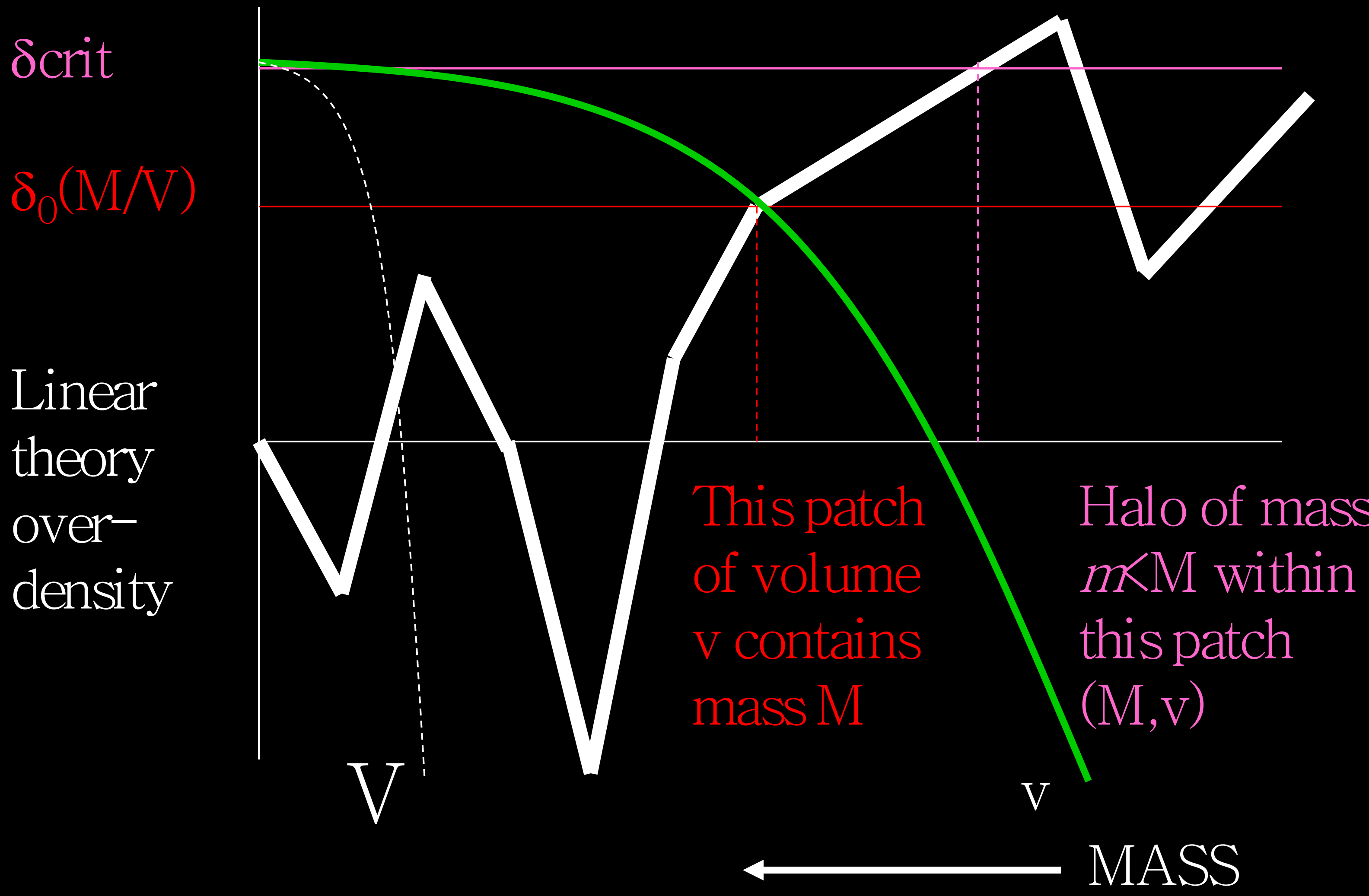
$$(R_{\mathrm{initial}}/R)^3 = \mathrm{Mass}/(\rho_{\mathrm{com}}\mathrm{Volume}) =$$

$$1 + \delta \approx (1 - \delta_0/\delta_{sc})^{-\delta_{sc}}$$

…which can be inverted:

$$(\delta_0/\delta_{sc}) \approx 1 - (1 + \delta)^{-1/\delta_{sc}}$$

The Nonlinear PDF



- Halo mass function is distribution of counts in cells of size $v \to 0$ that are not empty.

- Fraction f of walks which first cross barrier associated with cell size V at mass scale M,

$$f(M|V)\, dM = (M/V)\, p(M|V)\, dM$$

where $p(M|V)\, dM$ is probability randomly placed cell V contains mass M.

- Note: all other crossings irrelevant → stochasticity in mapping between initial and final density

- On large scales, barrier falls steeply from large height, so most walks which cross barrier do so only once. So no stochasticity, and

$$F(>M|V) = p[>\delta_{in}(M|V)|M] \quad \text{where}$$

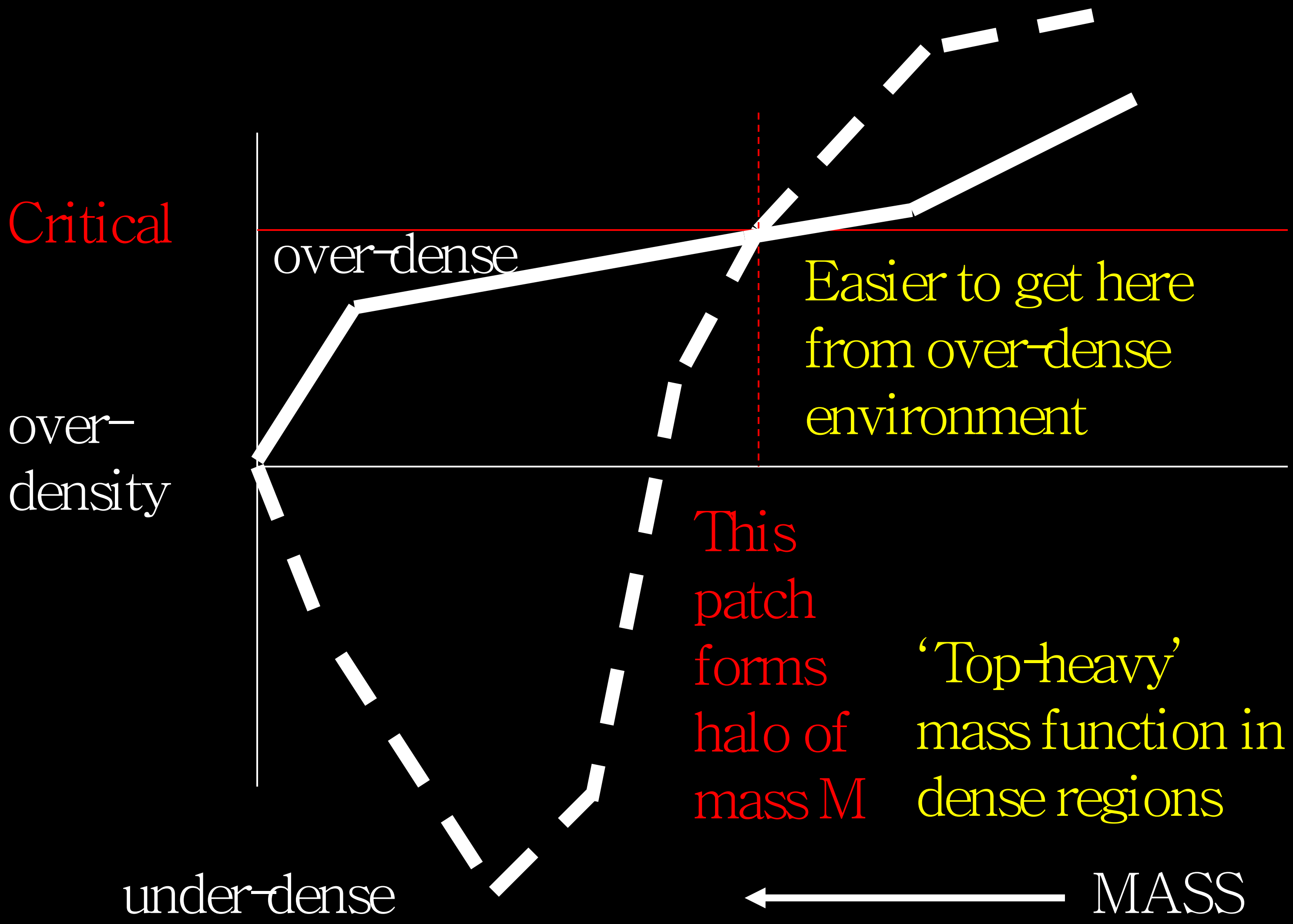
$$(\delta_{in}/\delta_c) = 1 - (M/\rho V)^{-1/\delta c}$$

- Provides estimate of late-time non-linear PDF if initial linear PDF known; shows why skewness is generic.

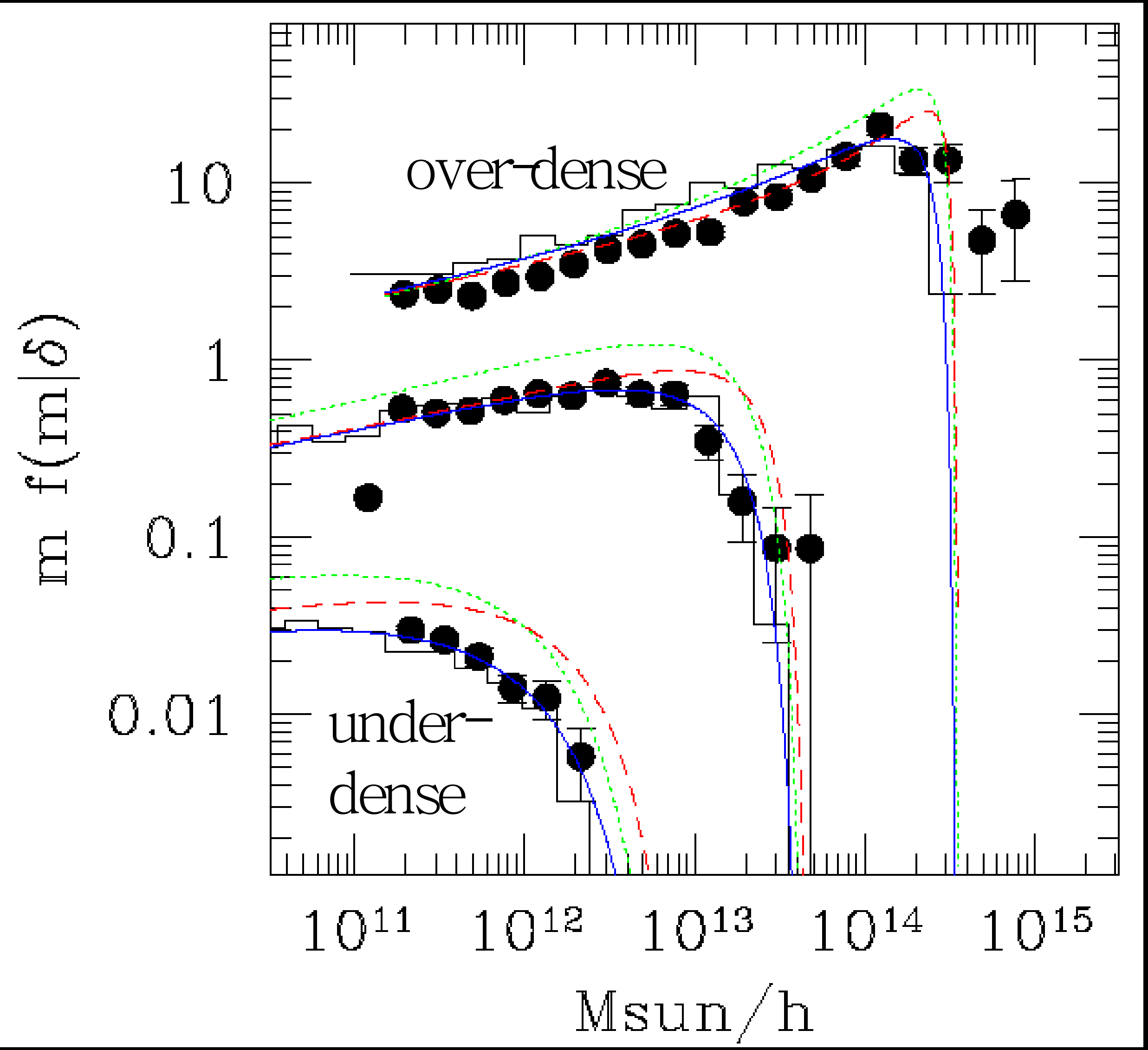
- For GRF, linear PDF Gaussian for all scales M; *not* true for most other PDFs, or for non-Gaussian ICs.

- This 'infinite-divisibility' of the Gaussian is also true of the Holtzmark distribution which features in Chandrasekhar's work on Dynamical Friction (sum of several $1/r^2$ vectors).

Correlations with environment



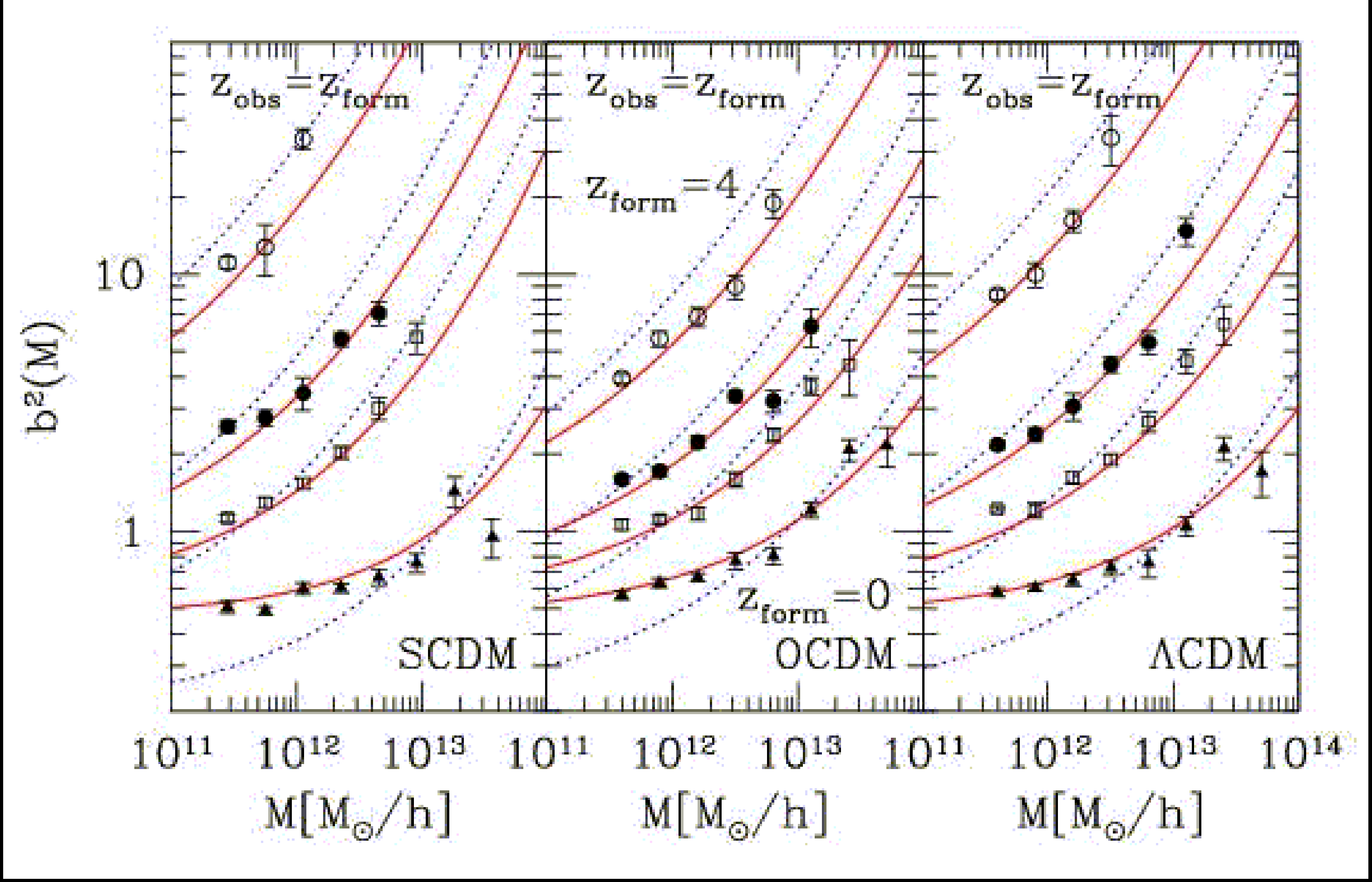
Most massive halos populate densest regions



Key to understand galaxy biasing (Mo & White 1996; Sheth & Tormen 2002)

$$n(m|\delta) = [1 + b(m)\delta]\, n(m) \neq [1 + \delta]\, n(m)$$

Halo clustering ← Halo abundances



Clustering also strong function of mass: can (should!) use clustering to calibrate mass

The Linear Barrier approximation

- $f(\delta_c, s)\, ds = (\delta_c^2/2\pi s)^{\frac{1}{2}}\ \exp(-B_c^2/2s)\, ds/s$

  $B_c = \delta_c\, (1 - \beta_V\, s/\delta_c^2)$

- $d\ln f/d\delta_c = 1 - bias$

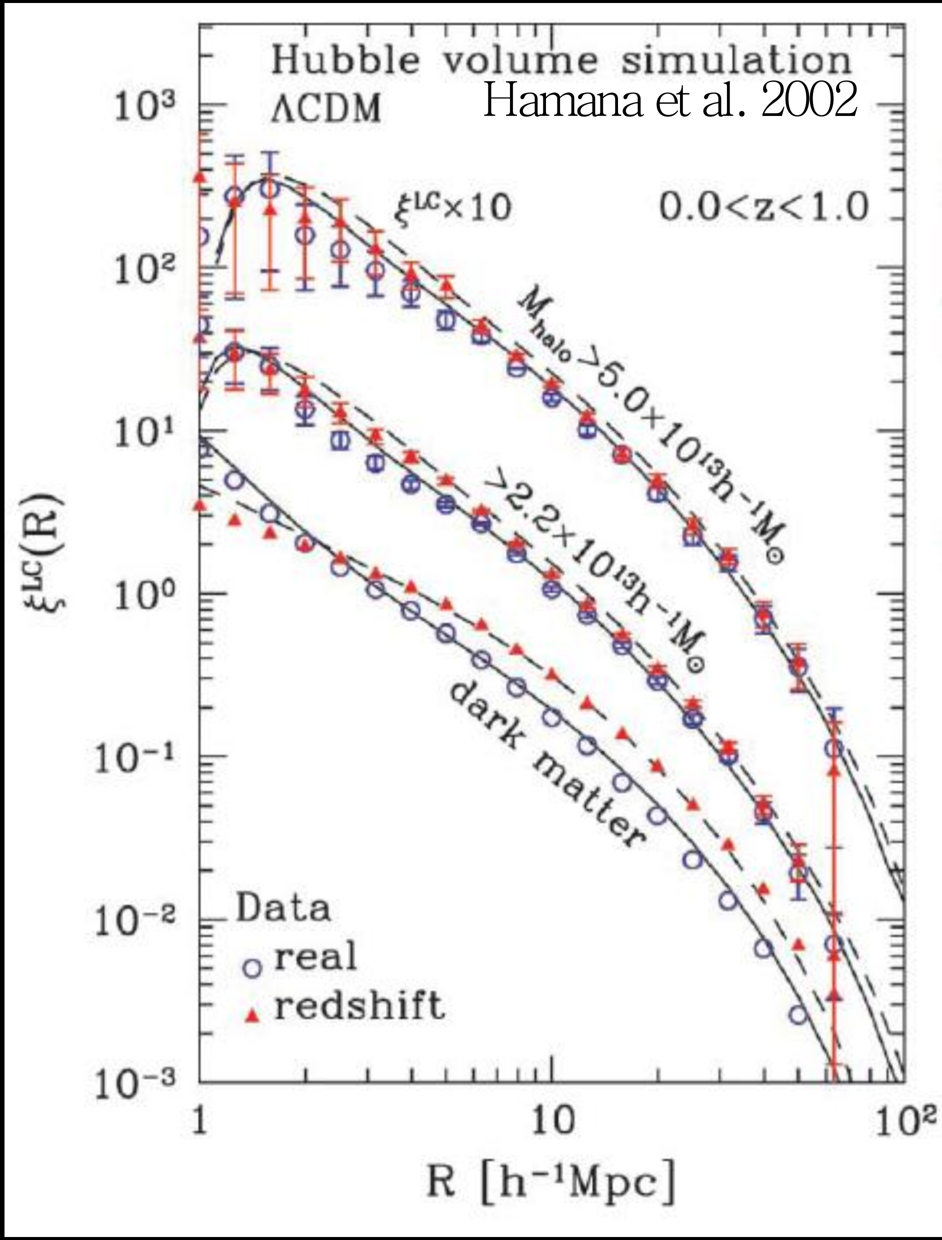
  $= 1/\delta_c - (B_c/s)\, (1 + \beta_V\, s/\delta_c^2)$

  $= 1/\delta_c - (\delta_c/s)\, [1 - \beta_V^2 (s/\delta_c^2)^2]$

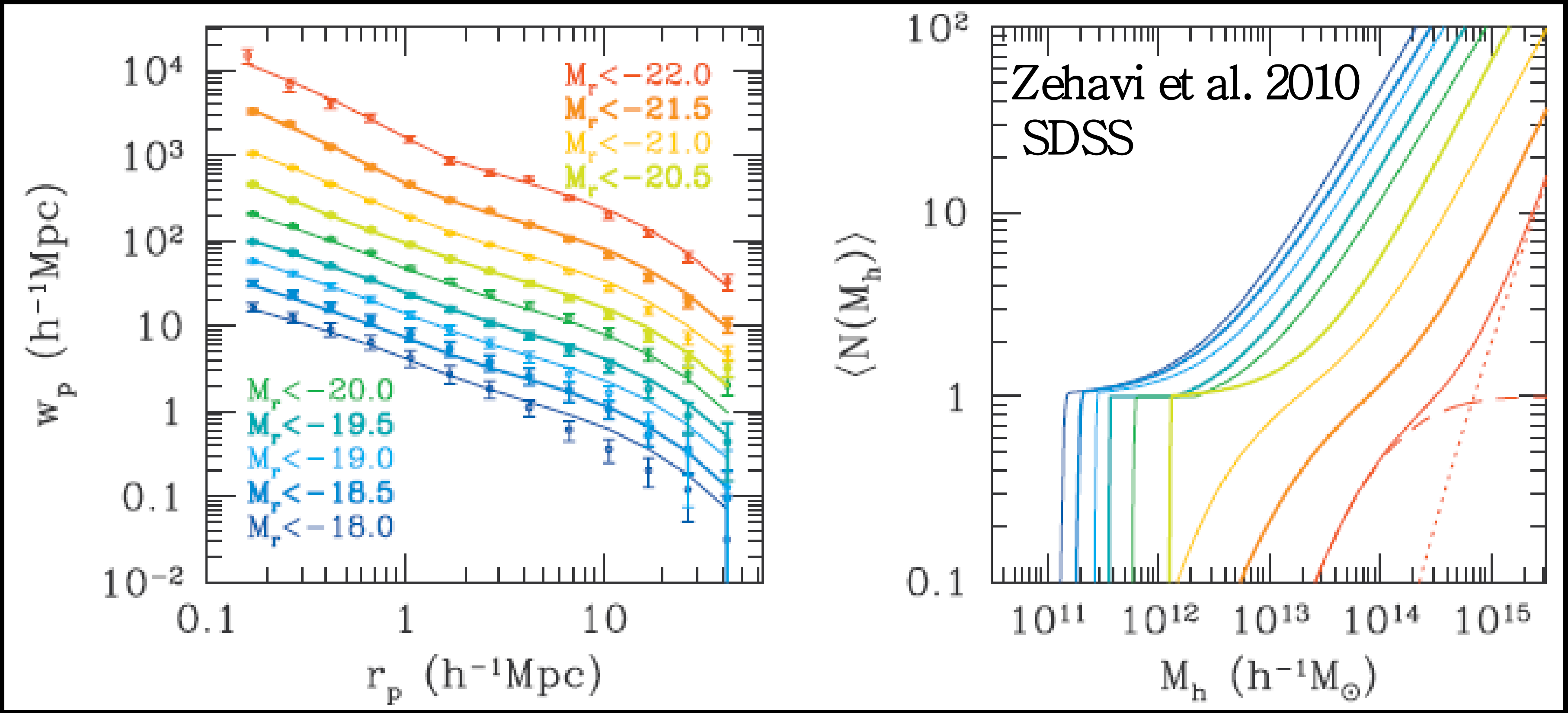
$bias = 1 + [(\delta_c^2/s) - 1]/\delta_c - \beta_V^2\, (s/\delta_c^2)/\delta_c$

Massive halos more strongly clustered

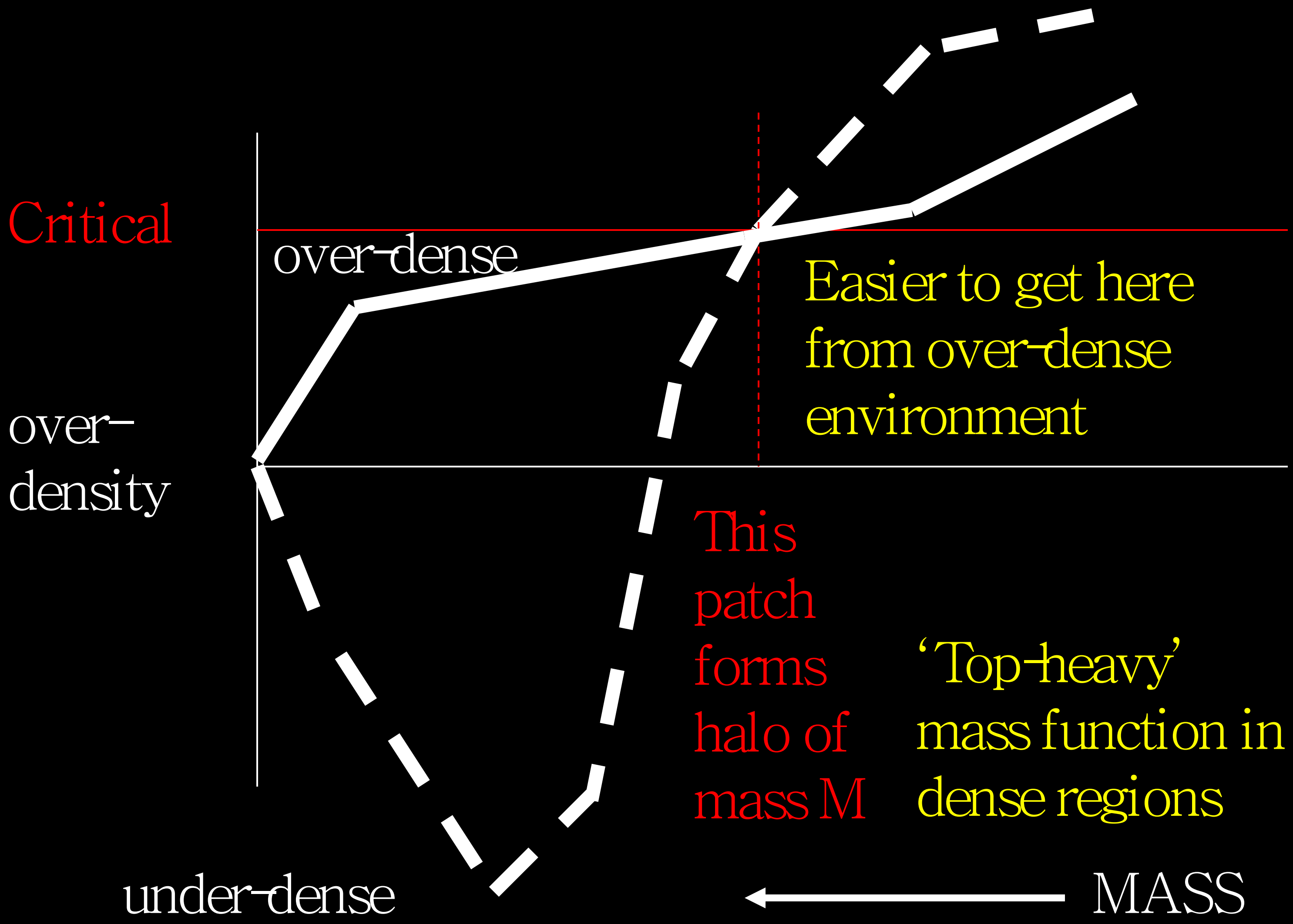
'linear' bias factor on large scales increases monotonically with halo mass



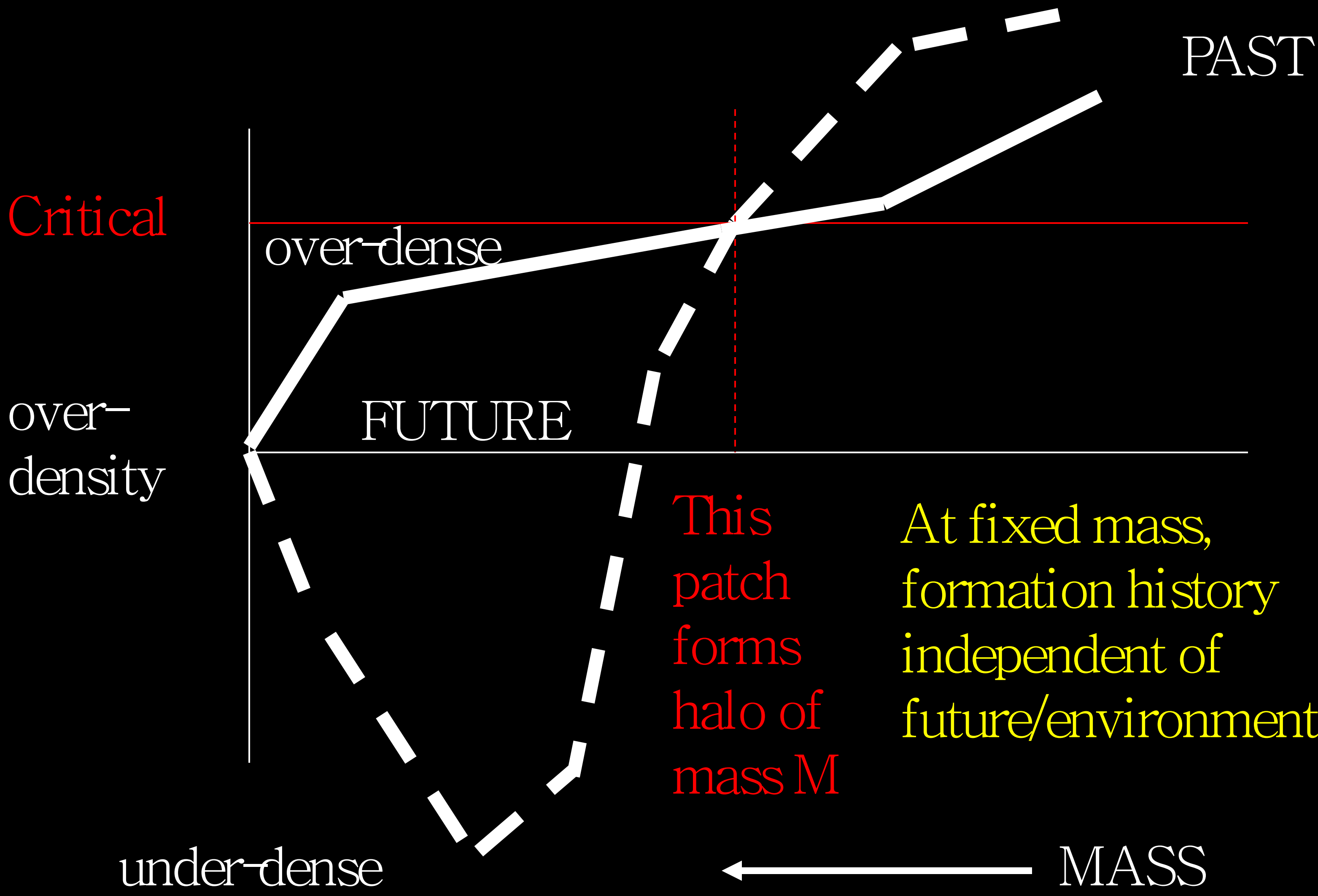
$$\langle N_{gal}|m\rangle = f_{cen}(m)\,[1 + \langle N_{sat}|m\rangle]$$
$$\approx 1 + m/15m_L$$



Correlations with environment



Correlations with environment

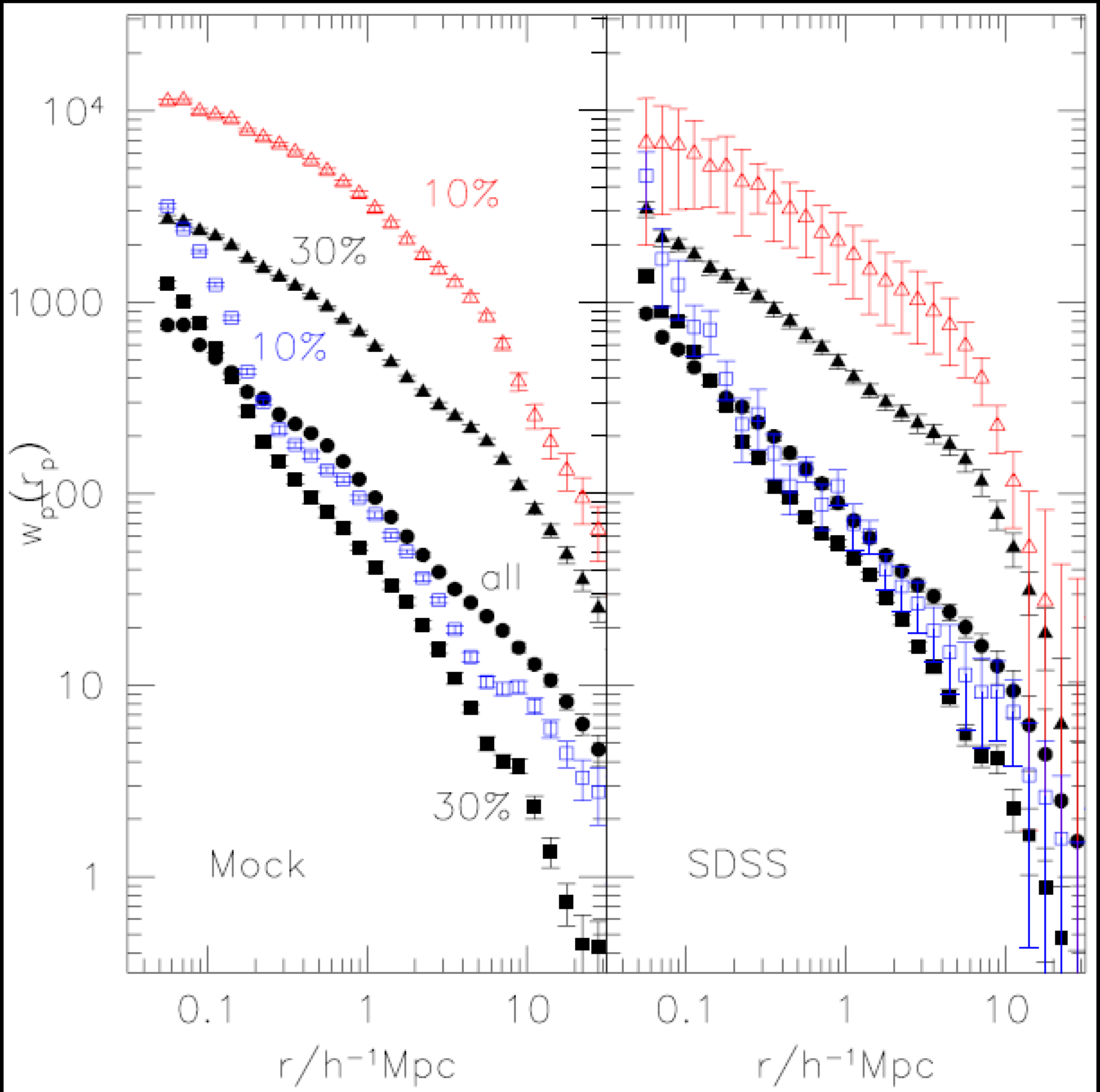


At fixed mass, formation history independent of future/environment

Environmental effects

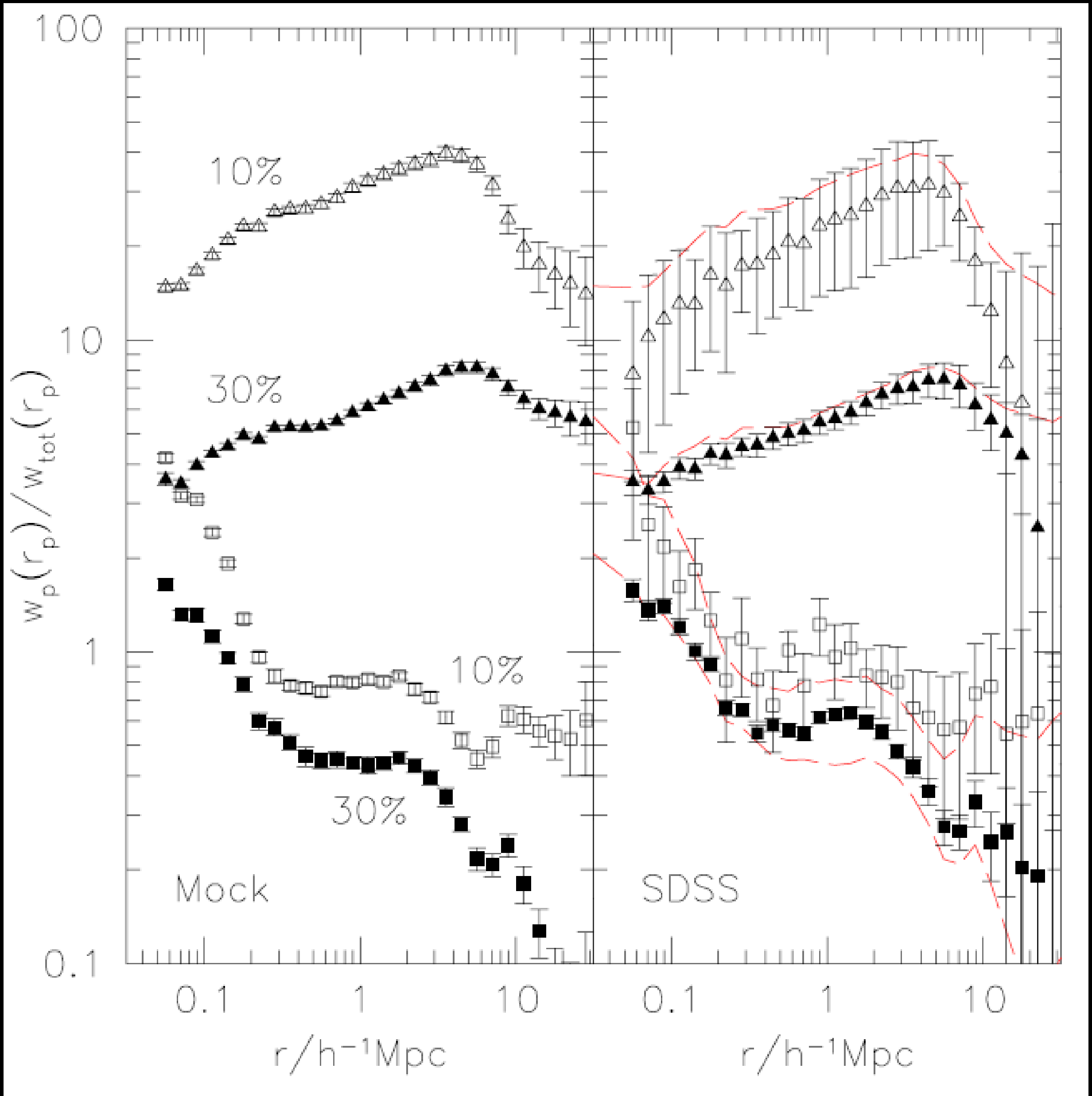
- Gastrophysics determined by formation history of parent halo
- All environmental trends come from fact that massive halos populate densest regions
- Greatly simplifies models of galaxy formation

- Environment = neighbours within 8 Mpc
- Clustering stronger in dense regions
- Dependence on density NOT monotonic in less dense regions!
- Same seen in mock catalogs; little room for extra effect!



Abbas & Sheth 2007

- Galaxy distribution remembers that, in Gaussian random fields, high peaks and low troughs cluster similarly



Broken symmetry between over- and under-dense regions can be understood from the Linear Barrier approximation

$$bias = 1 + [(\delta_c^2/s) - 1 - \beta_V^2 (s/\delta_c^2)]/\delta_c$$

The standard model:

- Can rescale halo abundances to 'universal' form, independent of *P(k)*, *z*, cosmology
  - Greatly simplifies likelihood analyses
  - Large voids are similar but less abundant, so require more volume to provide similar constraints
- Intimate connection between abundance and clustering of dark halos
  - Can use cluster clustering as check that cluster mass-observable relation correctly calibrated
  - Mass function top-heavy in dense regions
  - All environmental effects come from this correlation
- Unlikely to (do not!) hold at 1% precision

The future

- Excursion set models of morphology of large-scale structure
  - Requires barrier crossing by (at least) 6-dimensional walk
  - Chandrasekhar symmetry arguments work for barrier which is ‘sphere’; useful guide to progress
  - Model for dependence of halo bias on quantities other than mass
- Account for correlated steps + correlated walks

May this subject age as gracefully as did Chandrasekhar

You may think I have used a hammer to crack eggs, but I have cracked eggs.

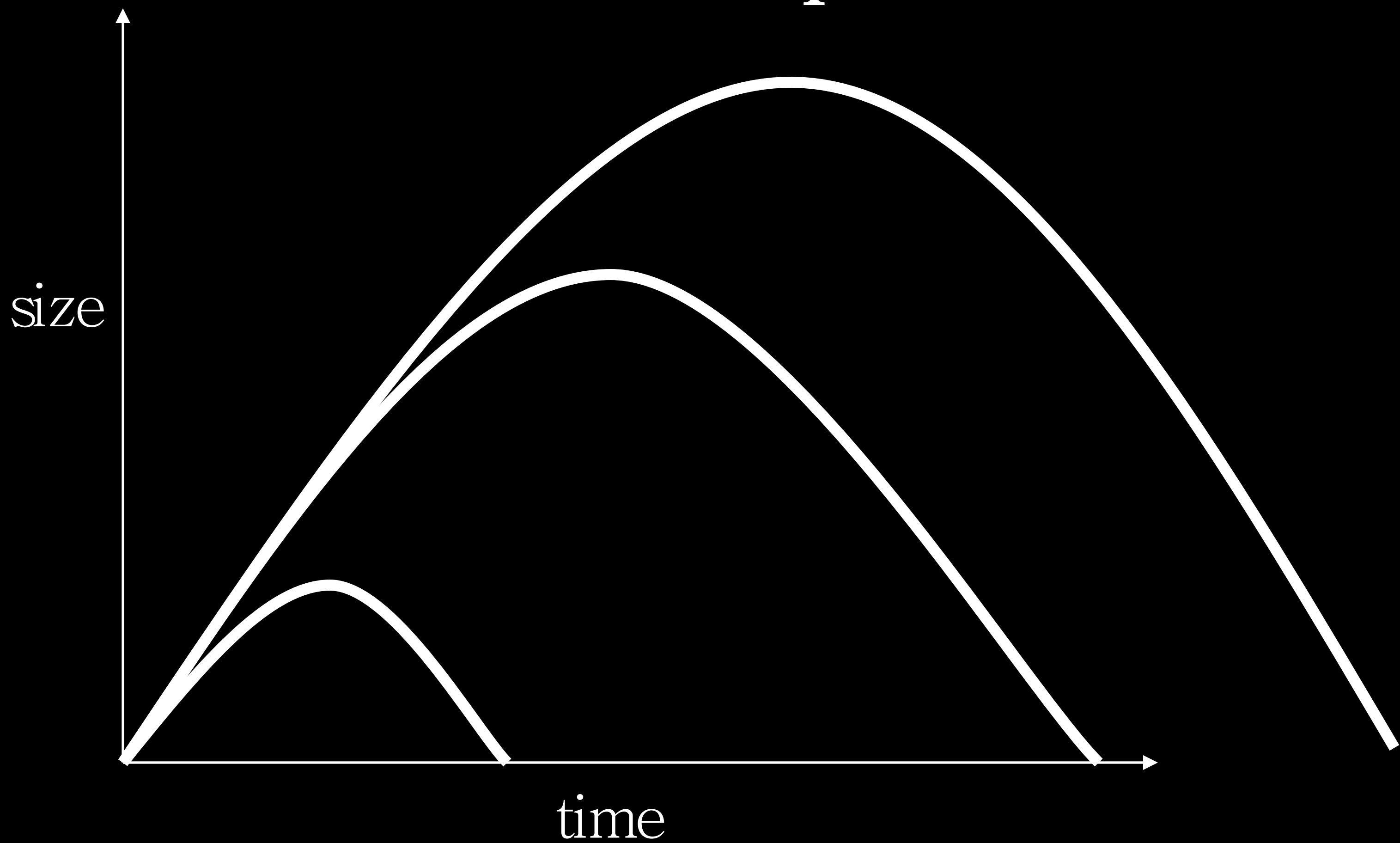
Assumptions of convenience

- Uncorrelated steps
  - Correlated steps (Peacock & Heavens 1990; Maggiore & Riotto 2009)
  - Non-Gaussian mass function ~ given by changing PDF but keeping uncorrelated steps
- Barrier is sharp
  - Fuzzy/porous barriers (Sheth, Mo, Tormen 2001; Maggiore & Riotto 2009)
- Ensemble of walks is also random
  - The real cloud-in-cloud problem

Only very fat cows are spherical….

(Lin, Mestel & Shu 1963; Icke 1973; White & Silk 1978; Bond & Myers 1996; Sheth, Mo & Tormen 2001; Desjacques 2008)

Triaxial collapse



Evolution of second axis very similar to spherical model of same initial density

So collapse of 1st axis sooner than in spherical model; collapse of all 3 axes takes longer

Tri-axial (ellipsoidal) collapse

- Evolution determined by properties of initial deformation field, described by 3×3 matrix at each point (Doroshkevich 1970)
- Tri-axial because 3 eigenvalues/invariants; Trace = initial density δ_{in} = quantity which determines spherical model; other two (e,p) describe anisotropic evolution of patch
- Critical density for collapse no longer constant: On average, $\delta_{ec}(\delta_{in}, e,p)$ larger for smaller patches → low mass objects

Convenient Approximations

- Zeldovich Approximation (1970):

$$(1+\delta)_{Zel} = \Pi_{i=1}^{3}\,(1-\lambda_i)^{-1}$$

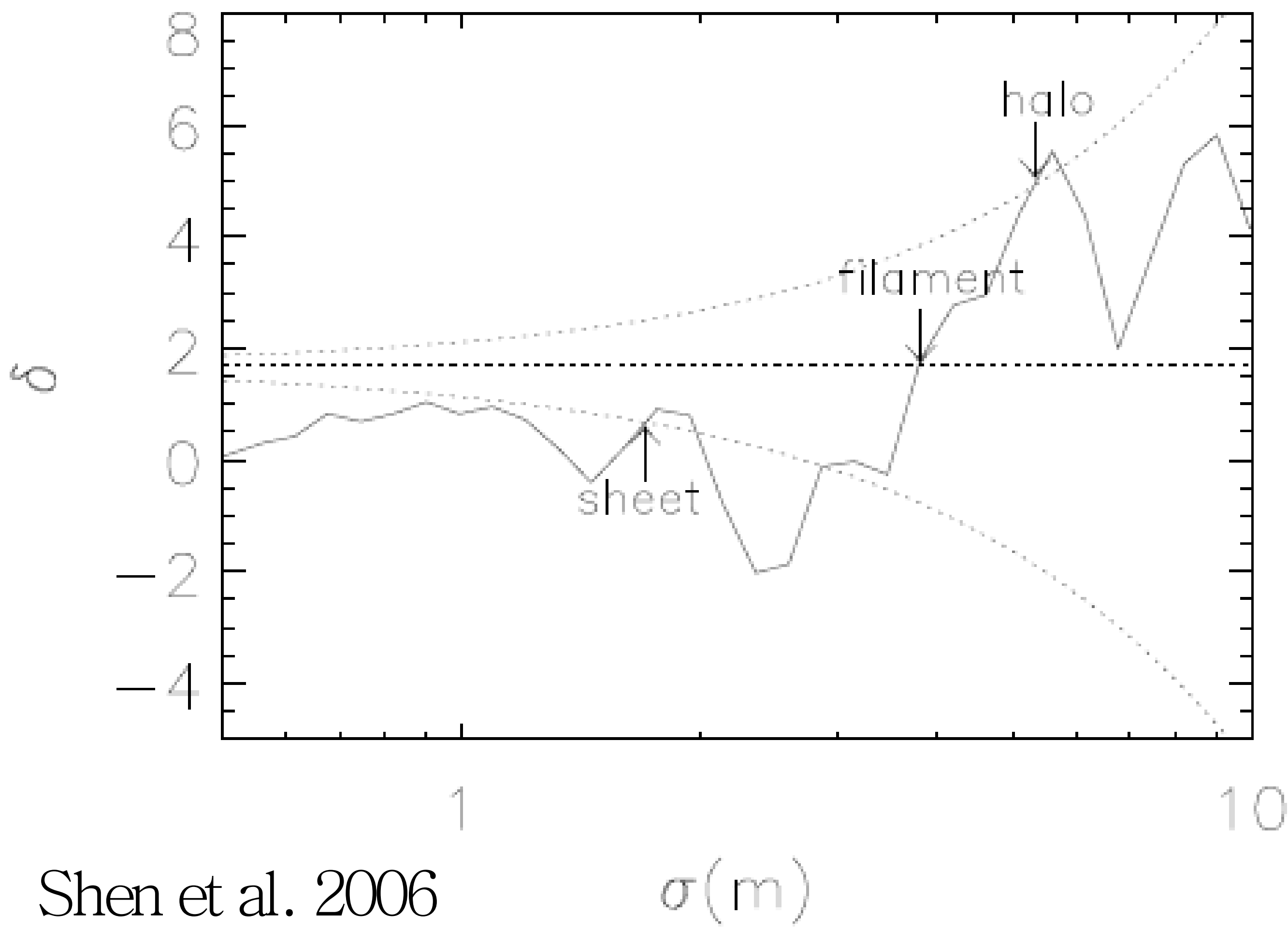
- Zeldovich Sphere ($\lambda_1 = \lambda_2 = \lambda_3 = \delta_{Linear}/3$):

$$(1+\delta)_{ZelSph} = (1-\delta_{Linear}/3)^{-3}$$

$$(1+\delta)_{EllColl} \approx$$

$$(1+\delta)_{SphColl}\,(1+\delta)_{Zel}/(1+\delta)_{ZelSph}$$

Triaxial collapse + excursion set description of the morphology of large scale structure



Shen et al. 2006

Halo abundances/galaxy formation care about morphology