MHD Turbulence

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Hydromagnetic turbulence. I. A deductive theory

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In this paper a deductive theory of turbulence recently described by the writer (Chandrasekhar 1955*a*) is extended to hydromagnetics. By considering stationary turbulence and making statistical hypotheses of the same general character as in the hydrodynamical theory, a pair of differential equations are derived for the scalars defining the isotropic tensors describing the correlation in the velocities and the intensities of the magnetic field at two different points and at two different times.

Hydromagnetic turbulence. II. An elementary theory

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In this paper an elementary theory of hydromagnetic turbulence is developed along the lines of Heisenberg's theory of ordinary turbulence. The basic physical idea underlying this theory is to conceive the transformation of the kinetic energy at a particular wave number into kinetic energy and magnetic energy at higher wave numbers, and similarly, the transformation of the magnetic energy at a given wave number into kinetic energy and magnetic energy of higher wave numbers, as a cascade process which can be visualized in terms of suitably defined coefficients of eddy viscosity and eddy resistivity. The resulting equations for the cascade process have been solved under stationary conditions in the limiting case of zero viscosity and infinite electrical conductivity. It is shown that in this limiting case there exist two distinct modes of turbulence; these have been distinguished as the velocity mode and the magnetic mode respectively. In both modes equipartition between the two forms of energy prevail among the largest eddies present (i.e. as $k \to 0$); and the spectrum of both the kinetic energy and the magnetic energy have a Kolmogoroff behaviour for $k \to 0$. The two modes differ in their behaviour for $k \to \infty$. In the velocity mode the ratio of the magnetic energy to the kinetic energy tends to zero among the smallest eddies present (i.e. as $k \to \infty$), while in the magnetic mode the same ratio tends to about 2·6 as $k \to \infty$. The bearing of these results on the possible character of the interstellar magnetic fields is briefly discussed.

- Theories of Iroshnikov (1963), Kraichnan (1965)
- Interstellar Scintillation data (since 1968) does not agree with IK theory
- New theories in the 1990s -> applications in many areas of astrophysics
- In situ spacecraft measurements of Solar Wind Turbulence -> generalizations in the 2000s

MHD Turbulence

- Strong background magnetic field: turbulence due to nonlinear interactions between oppositely directed Alfven waves
- Alfven wave turbulence
  - Highly anisotropic
  - Weak or strong
  - Balanced or imbalanced

Alfven waves (incompressible fluid)

- Mean magnetic field $\mathbf{B} = B_0 \hat{z}$
- Both $\delta\mathbf{v}$ and $\delta\mathbf{B}$ are parallel to $\mathbf{k} \times \hat{z}$
  (slow waves are the other orthogonal component)
  - Up waves

$$\delta\mathbf{v} = -\frac{\delta\mathbf{B}}{\sqrt{4\pi\rho}}$$

$$\omega^{+}(\mathbf{k}) = +V_A\, k_z$$

Down waves

$$\delta\mathbf{v} = +\frac{\delta\mathbf{B}}{\sqrt{4\pi\rho}}$$

$$\omega^{-}(\mathbf{k}) = -V_A\, k_z$$

$$V_A = \frac{B_0}{\sqrt{4\pi\rho}}\,; \qquad \text{Alfven speed}$$

- Elsasser fields

$$\mathbf{w}^{\pm} = \delta\mathbf{v} \mp \frac{\delta\mathbf{B}}{\sqrt{4\pi\rho}}$$

- Nonlinear Waves: when either $\mathbf{w}^{+}$ or $\mathbf{w}^{-}$ is $\mathbf{0}$, the nonlinear terms vanish.
  Exact solutions:
  - Up waves: $\mathbf{w}^{+} = \mathbf{f}^{+}(x, y, z - V_A t)\,, \qquad \mathbf{w}^{-} = \mathbf{0}$
  - Down waves: $\mathbf{w}^{-} = \mathbf{f}^{-}(x, y, z + V_A t)\,, \qquad \mathbf{w}^{+} = \mathbf{0}$

Processes underlying MHD Turbulence

- MHD Turbulence is the result of nonlinear interactions between oppositely directed Alfven wavepackets
  Iroshnikov (1963), Kraichnan (1965)

- Energy conservation:

$$E^{\pm} = \int \frac{|\mathbf{w}^{\pm}|^2}{2}\, d^3x\,; \quad \text{Elsasser energies}$$

- No exchange between E^{+} and E^{-}

- Collisions only lead to a redistribution of energies over different length scales

Iroshnikov-Kraichnan theory

- Iroshnikov (1963), Kraichnan (1965)
- Balanced isotropic excitation of +ve and -ve waves on length scale $\sim L$ and rms amplitudes $w_L < V_A$
- Assume isotropy on all scales $\lambda < L$
- Each collision perturbs a wavepacket weakly:

$$\delta w_\lambda \sim \frac{w_\lambda^2}{V_A}$$

- Successive collisions add with random phases. The number of collisions for perturbations to build up to order unity is

$$N_\lambda \sim \left(\frac{V_A}{w_\lambda}\right)^2$$

- The cascade time is

$$t_\lambda \sim N_\lambda \frac{\lambda}{V_A}$$

- The λ-independence of

$$\varepsilon \sim \frac{w_\lambda^2}{t_\lambda}$$

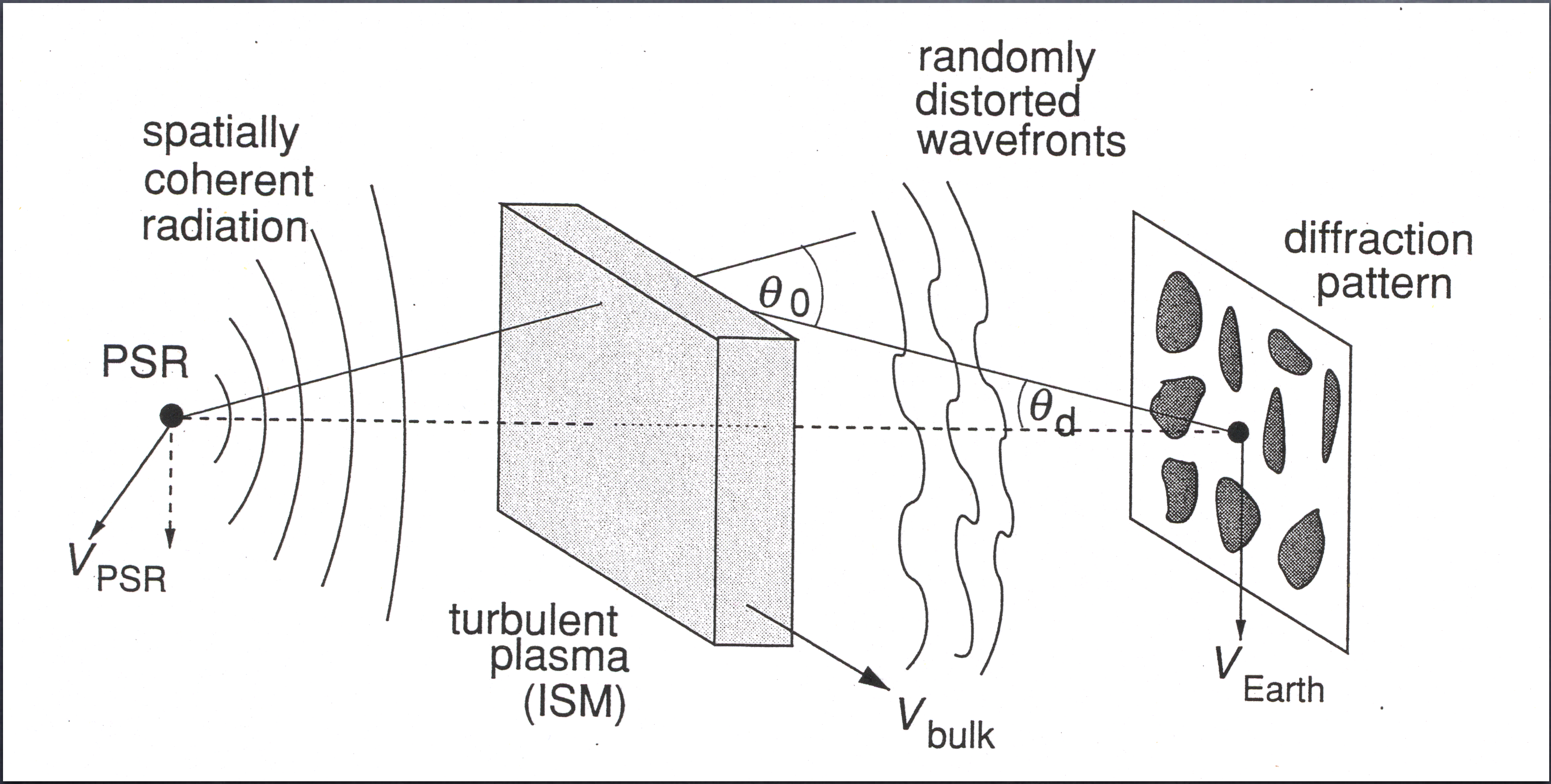
- implies that in the inertial-range

$$w_\lambda \;\sim\; w_L \left(\frac{\lambda}{L}\right)^{1/4}$$

- Can check that

$$N_\lambda \propto \lambda^{-1/2}$$

- So the IK theory seems self-consistent



Interstellar Scintillation

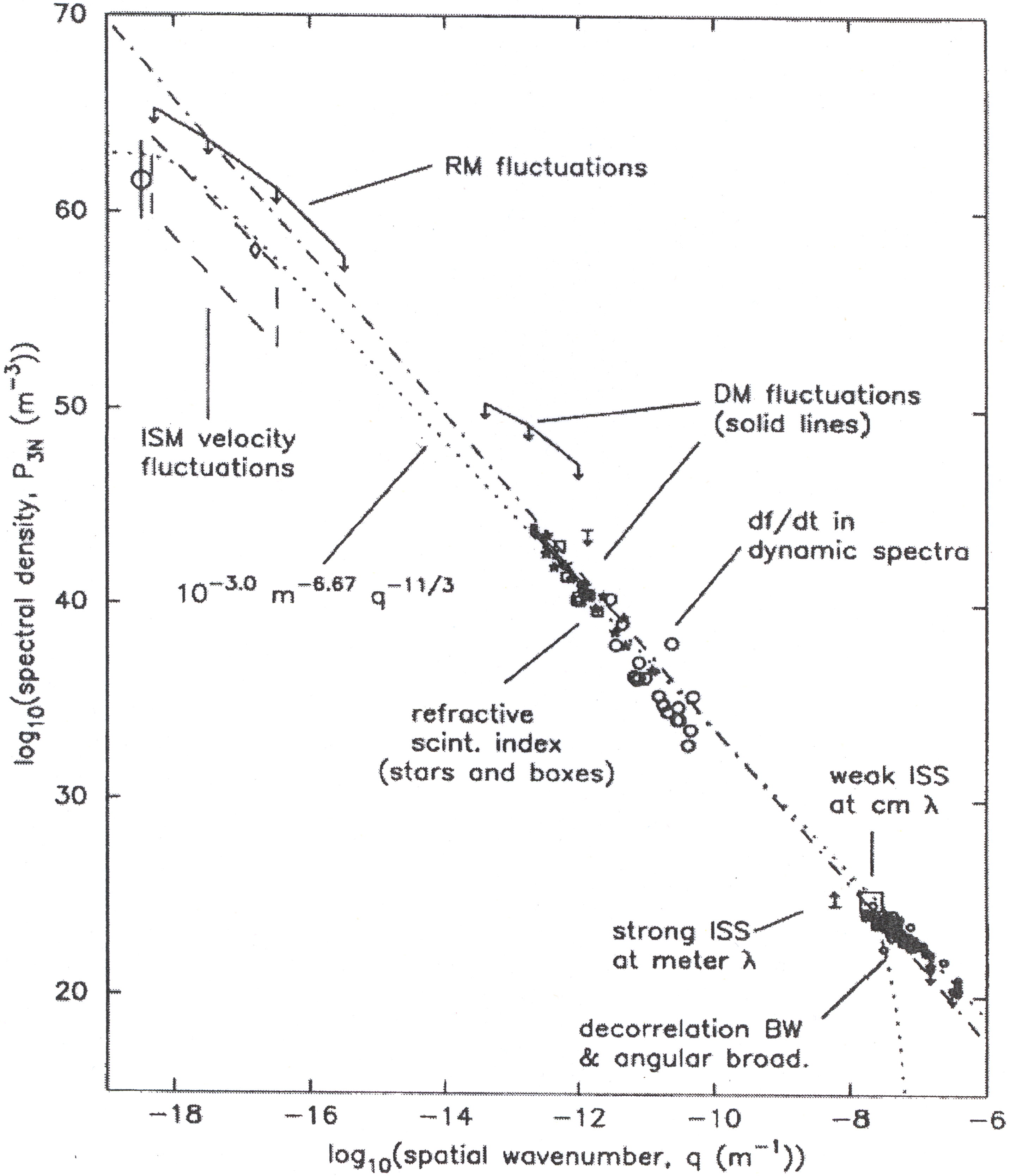


Figure 8: Power spectrum of electron density fluctuations, from Armstrong, Rickett, and Spangler, 1995, ApJ, **443**, 209

Interstellar Turbulence

- Electron density fluctuations follow a Kolmogorov spectrum
- If these result from mixing by a turbulent velocity field, then the velocity fluctuations must follow a Kolmogorov spectrum
- But hydrodynamic turbulence has a large inner scale!
- MHD turbulence? Higdon (1984)
- But the IK theory predicts a flatter spectrum!

Problems with the IK theory

- The IK theory fails because of the neglect of the 3-wave resonance conditions

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 \,, \qquad \omega_1^{\pm} + \omega_2^{\mp} = \omega_3^{\pm}$$

$$\text{where} \quad \omega_k^{\pm} = \pm V_A k_z$$

- One of k_{1z} or k_{2z} must be zero
Shebalin, Matthaeus & Montgomery (1983)

- Hence waves with values of k_z not present initially cannot be created during wavepacket collisions

- No parallel cascade
- Weak MHD turbulence must be anisotropic
- Balanced weak 3-wave turbulence
  Ng & Bhattacharjee (1996), Goldreich & SS (1997):

$$\text{Parallel scale} \sim L\,, \qquad \text{Transverse scale} \sim \lambda$$

$$w_\lambda \propto \lambda^{1/2}\,, \qquad N_\lambda \propto \lambda$$

- Theory must break down when $N_\lambda \sim 1$

Balanced Strong MHD Turbulence

- Goldreich & SS (1995)

- Critical Balance: $N_\lambda \sim 1$ is independent of λ
  - Cascade time is equal to the wave period
  - Resonance conditions do not hold
  - Parallel length scales can change: Λ_λ

$$t_\lambda \sim \frac{\lambda}{w_\lambda} \sim \frac{\Lambda_\lambda}{V_A}$$

- The λ-independence of $\varepsilon \sim w_\lambda^2/t_\lambda$ leads to the anisotropic Kolmogorov spectrum

$$w_\lambda \sim w_L \left(\frac{\lambda}{L}\right)^{1/3}$$

- Parallel Cascade

$$\Lambda_\lambda \sim \frac{V_A}{w_L} L^{1/3} \lambda^{2/3}$$

Comments

- Theory non perturbative; phenomenological
- Many applications: accretion discs, cosmic rays...
- Many outstanding problems in Interstellar Turbulence
    - Correlations between DM and SM
    - Steep spectra? Discrete structures?
    - Extreme scattering events
  - What are the sources and sites?
- But Interstellar Scintillation is not a direct probe!

Solar Wind Turbulence

- In situ spacecraft measurements since the 1960s
- Fast solar wind: speed $\sim 750\,\mathrm{km\,s^{-1}}$
- Slow solar wind: speed $\sim 350\,\mathrm{km\,s^{-1}}$
- Fast wind more uniform and steady
  - Solar minimum
  - High latitudes
- Ulysses spacecraft (October 1990 - June 2009)

Spacecraft measurements

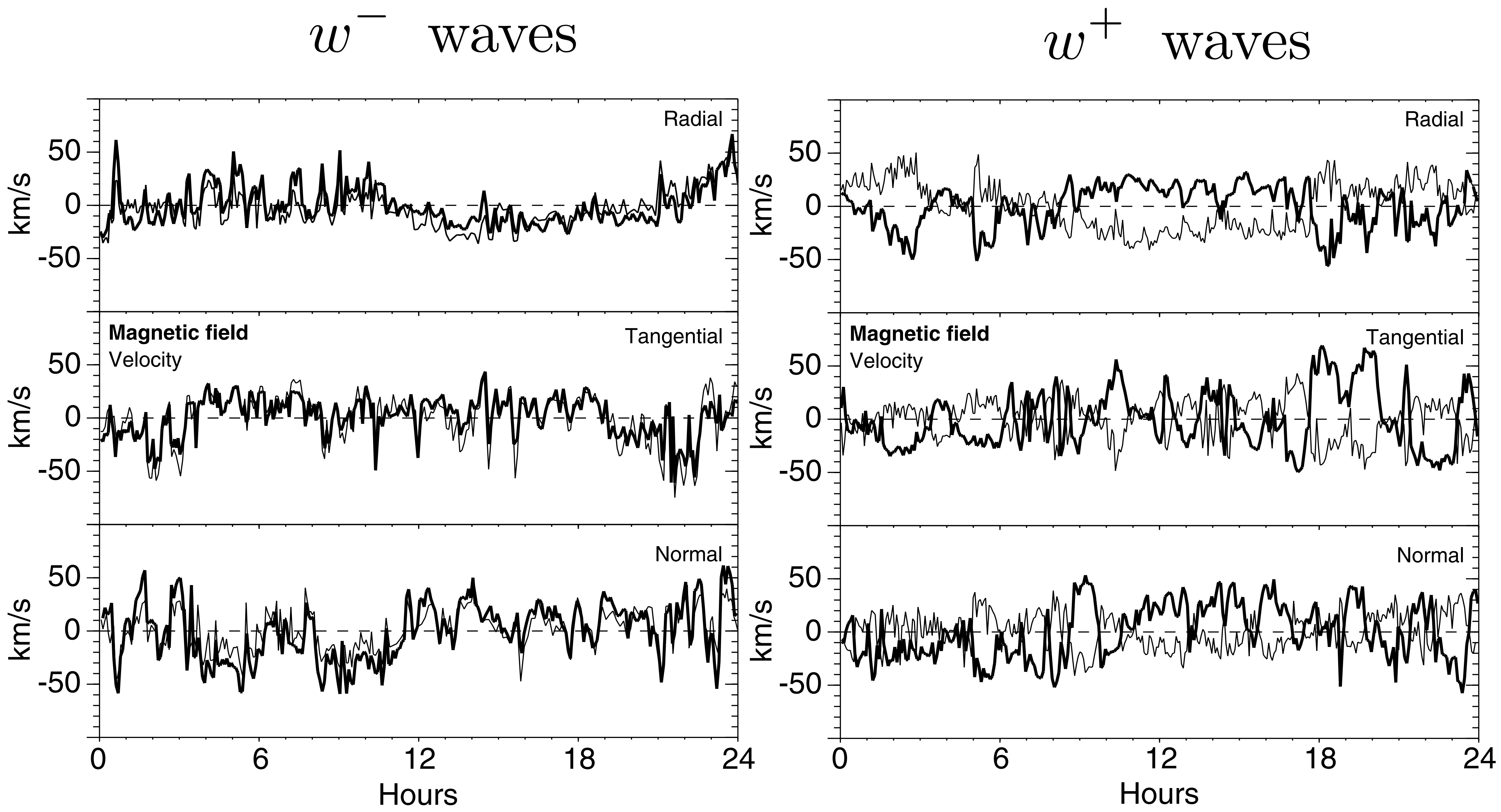
- Spacecraft time series is a straight line cut which is like a snapshot of the solar wind plasma
- Taylor's hypothesis: $k_0 = \frac{2\pi f}{V}$
- Reduced spectrum

$$P_{ij}^{\mathrm{red}}(f) = \int \mathrm{d}^3 k \, \delta(\mathbf{k} \cdot \hat{\mathbf{V}} - k_0) \, P_{ij}(\mathbf{k})$$

- Full recovery possible if 3-dim spectrum is isotropic
- Balanced strong MHD turbulence

$$P_{ij}^{\mathrm{red}}(f) \propto \begin{cases} f^{-5/3} & \text{if } \mathbf{V} \perp \mathbf{B} \\ f^{-2} & \text{if } \mathbf{V} \| \mathbf{B} \end{cases}$$

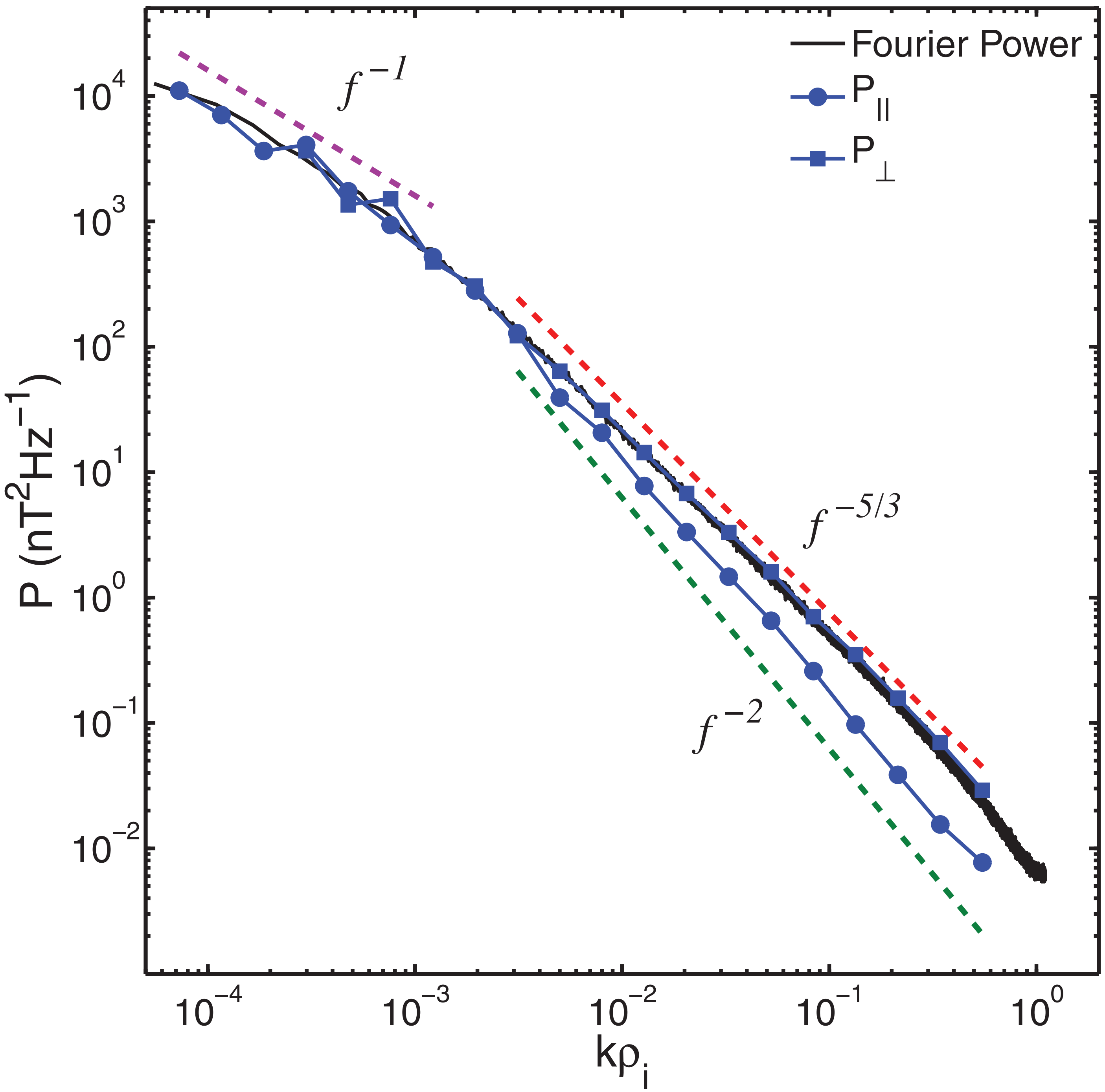
Alfven waves (Belcher & Davis 1971)



In both cases the Alfven waves are outward-bound

Polarization: $\delta \mathbf{v}$ and $\delta \mathbf{B}$ perpendicular to $\mathbf{B}$

Spatial variation: mostly transverse to $\mathbf{B}$



Wicks et al (2010): consistent with GS95 theory

Imbalanced Strong MHD Turbulence

- Lithwick, Goldreich & SS (2007)
- Anisotropic Kolmogorov spectra:

$$w_\lambda^\pm \sim \frac{(\varepsilon^\pm)^{2/3}}{(\varepsilon^\mp)^{1/3}} \lambda^{1/3}$$

$$\Lambda_\lambda \sim \frac{(\varepsilon^-)^{1/3}}{(\varepsilon^+)^{2/3}} V_A \lambda^{2/3}$$

- The ratio of the Elsasser amplitudes = the ratio of the corresponding energy fluxes
- Modified critical balance:
  cascade time = correlation time of straining imposed by oppositely directed waves
- When the energy fluxes are equal, the turbulence corresponds to the balanced strong cascade

- Other views on Imbalanced MHD Turbulence
  - Beresnyak & Lazarian (2008)
  - Chandran (2008)
  - Perez & Boldyrev (2009)
  - Podesta & Bhattacharjee (2010)
- The Solar Wind is probably the best laboratory we have to study MHD Turbulence