Chance & Chandra

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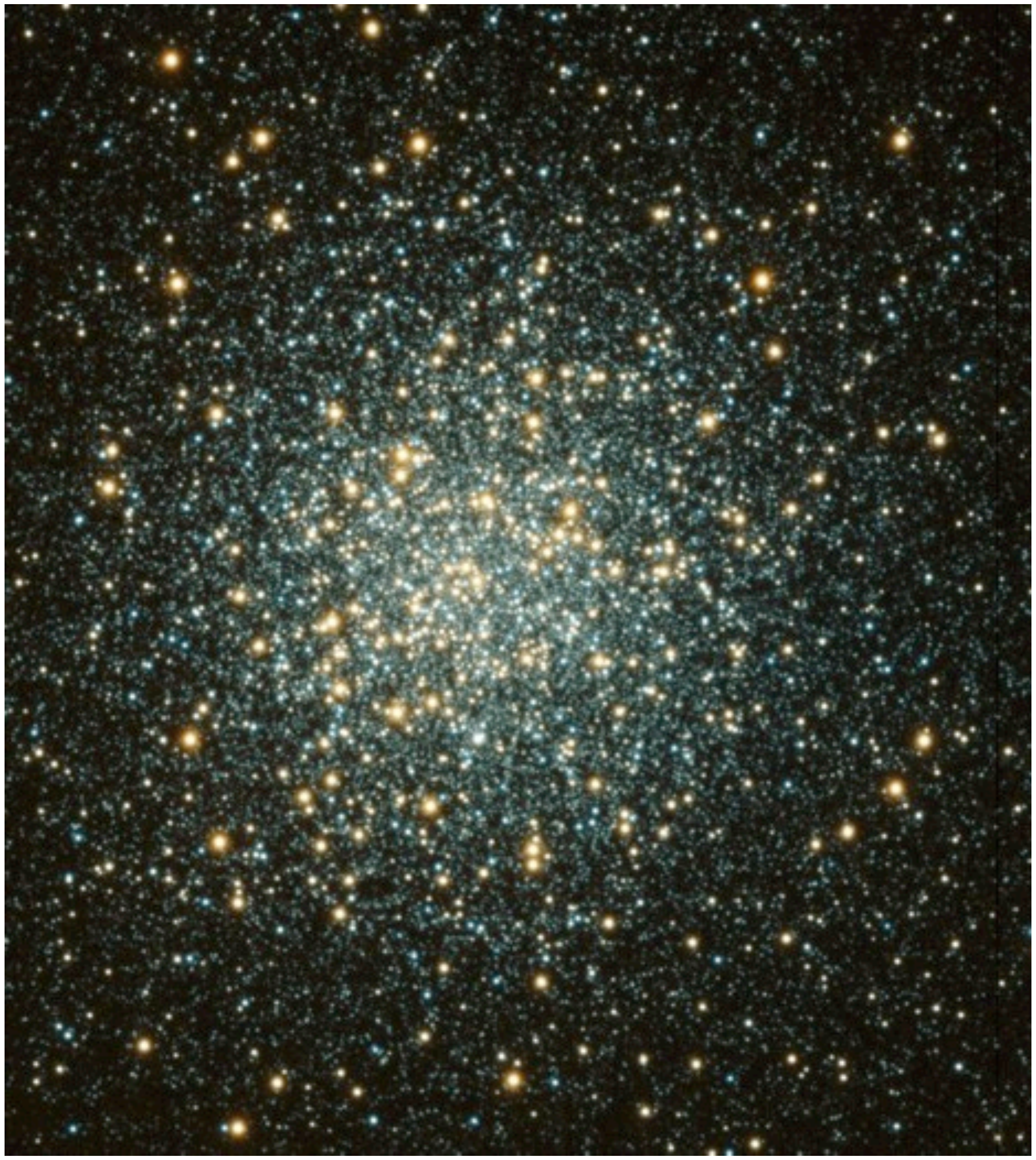
Chandrasekhar's contributions to statistical astrophysics

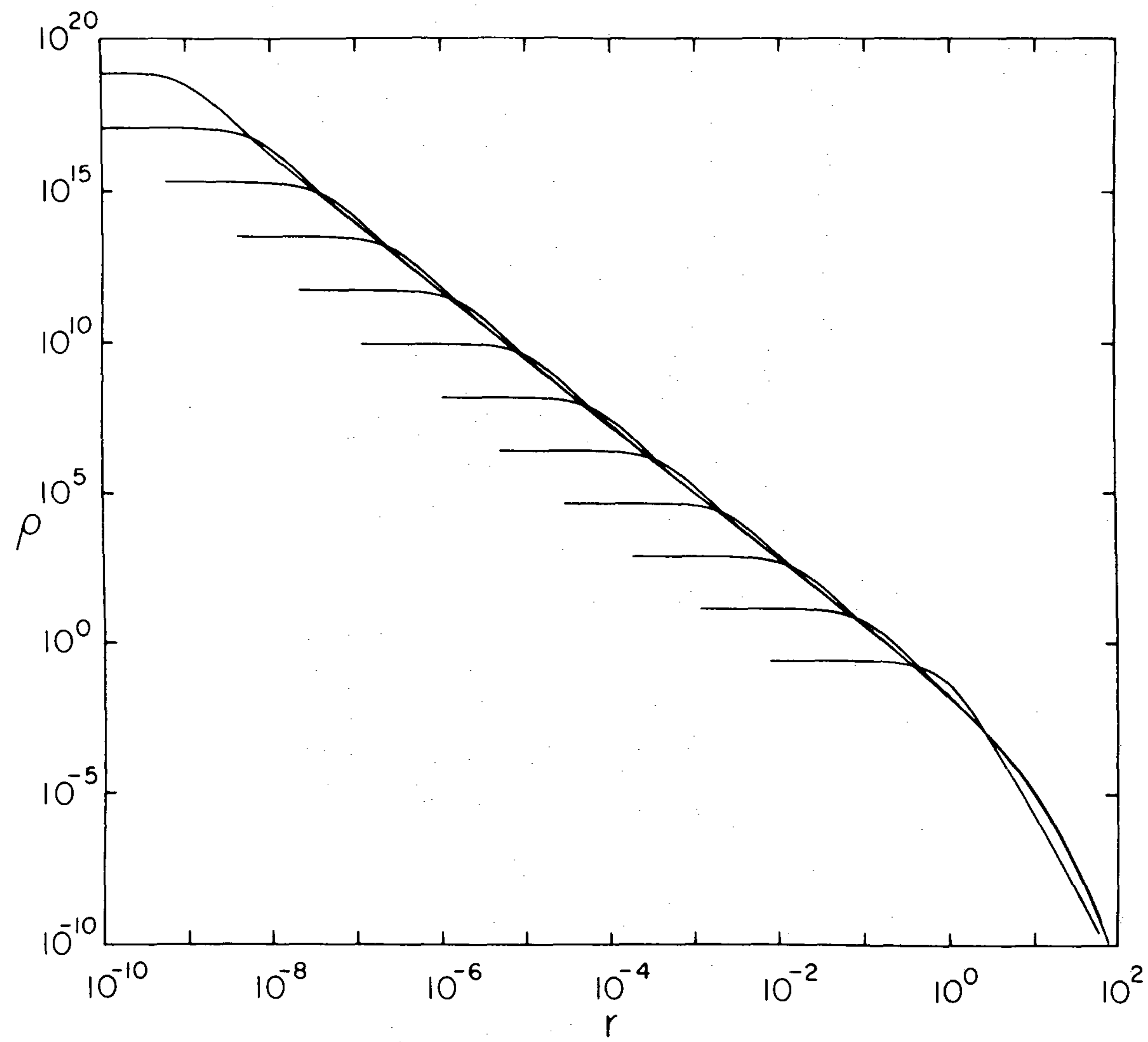
- Statistical inference
  - Milky Way brightness fluctuations (w. Münch)
  - distrib. of stellar spins from observed *vsini*
- Stochastic processes
  - turbulence
  - scintillation
  - random walks (Rev Mod Phys article)
  - **dynamical relaxation**

Chandra's relaxation

- Probability distribution of instantaneous force
  - & probabilities of durations of a given force
- Other applications of these methods
  - Stark broadening
  - gravitational microlensing
  - cosmological peculiar accelerations
- Dynamical friction from requirement of Maxwellian equilibrium (1943)

Dynamical relaxation in star clusters





H. Cohn 1980

Relaxation and chaos

- Exponential vs. algebraic evolution
- Resonances vs. encounters
  - Relaxation assumes uncorrelated encounters
  - chaos tends to promote this, but not completely
    - resonant relaxation: Rauch & Tremaine 1996
- Some systems require both languages
  - populous planetary systems
  - sparse star clusters with central SMBH/IMBH

Relaxation, thermodynamics, and statistical mechanics

- Maxwellian velocity distributions, approx.
- ``Specific heats'' of self-gravitating systems
- But newtonian stellar dynamics lacks important pre-requisites for true thermodynamics
  - energies are not bounded from below
  - energies are not extensive
- Plasma physics does have these, despite inverse-square forces, thanks to quantum mechanics & charge screening

Dark matter

- ``Standard'' dark matter is produced thermally, but is nearly collisionless today
- Weak collisionality of DM with itself is sometimes hypothesized to solve apparent difficulties on small scales
  - e.g. Spergel & Steinhardt 2000; Loeb & Weiner 2010
- Collisionality risks gravothermal collapse
- What would it take to prevent collapse and enable true thermodynamics?

Standard dark matter

- Cold
- Collisionless
- WIMPy:

$$\Omega_{\mathrm{WIMP}}\sigma_{\mathrm{annihilation}} \approx 10^{-36}\left[1 \; + 0.05\ln\left(\frac{m_{\mathrm{WIMP}}c^2}{\mathrm{GeV}}\right)\right]^{3/2} \mathrm{cm}^2$$

...presuming an initially thermal abundance with negligible chemical potential.

Advantages of SDM

- Consistent with needs of high-energy physics
  - lightest supersymmetric particle(?)
  - experimentally detectable (!?)
- Very simple astrophysics
  - collisionless, gravitating point masses
    - amenable to N-body simulation

Entia non sunt multiplicanda sine necessitatem.

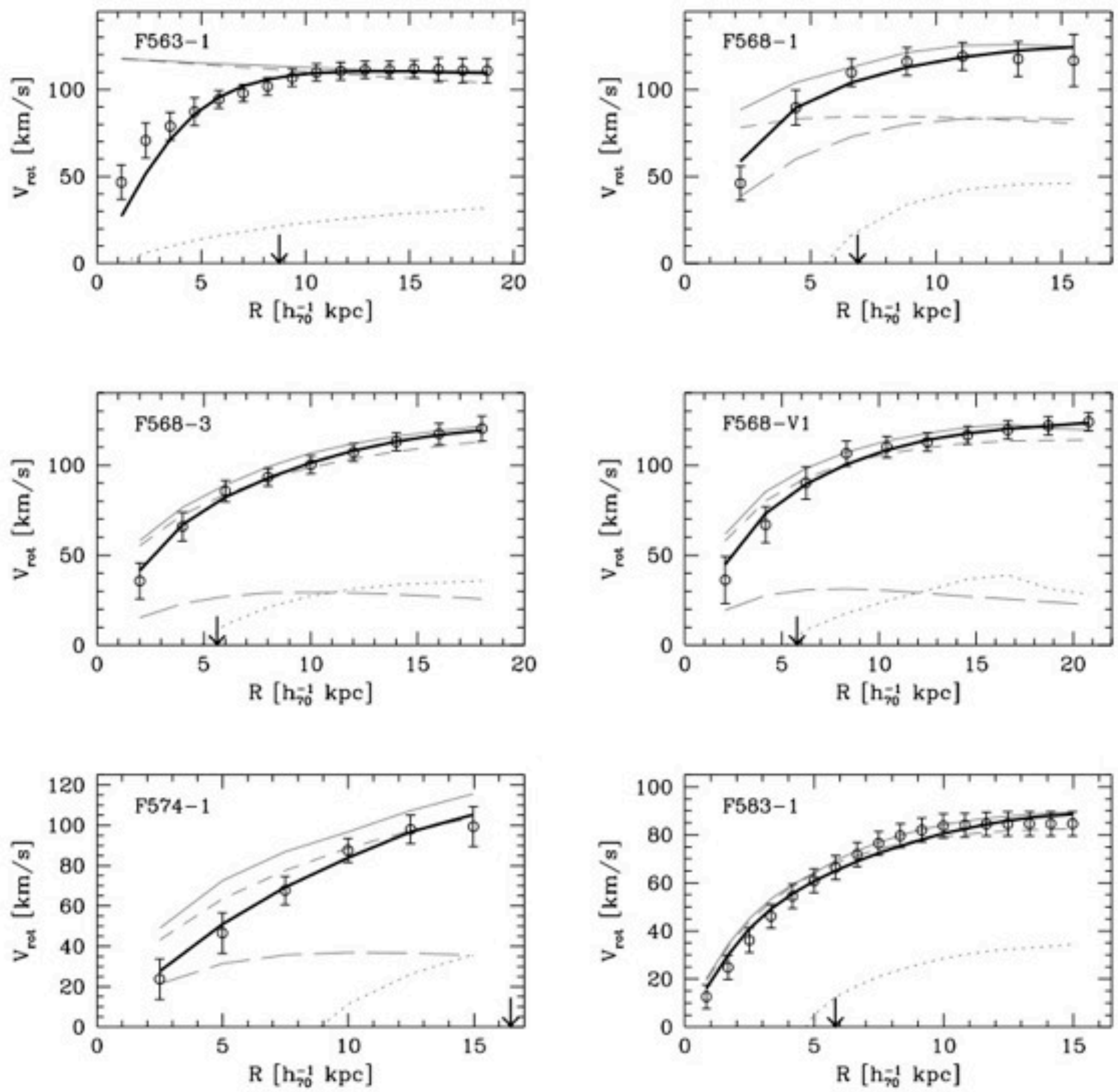
Worries

- Small-scale smoothness/lack of substructure
  - few small satellites of major galaxies
    - survival of thin disks
  - ``cored'' rotation curves of dwarf galaxies
- Abundance and rotation rate of galactic bars
    - Debattista & Sellwood 1998, 2000
- Conspiracy between dark and visible matter to produce isothermal mass profiles: $\rho_{\rm total} \propto r^{-2}$
- Sizes and angular momenta of spiral disks

Subhalos

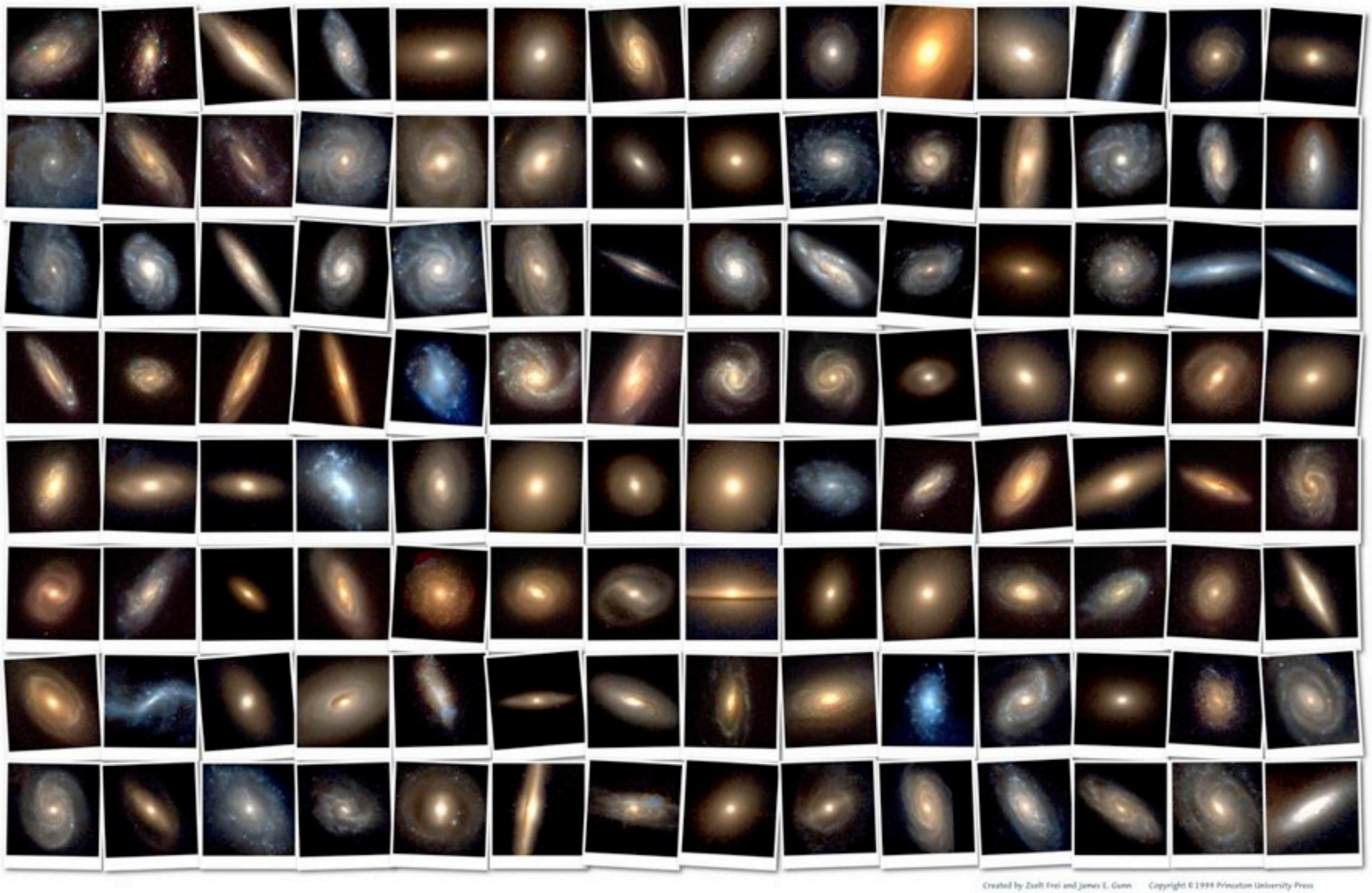
Diemand et al. 2008

Rotation curves



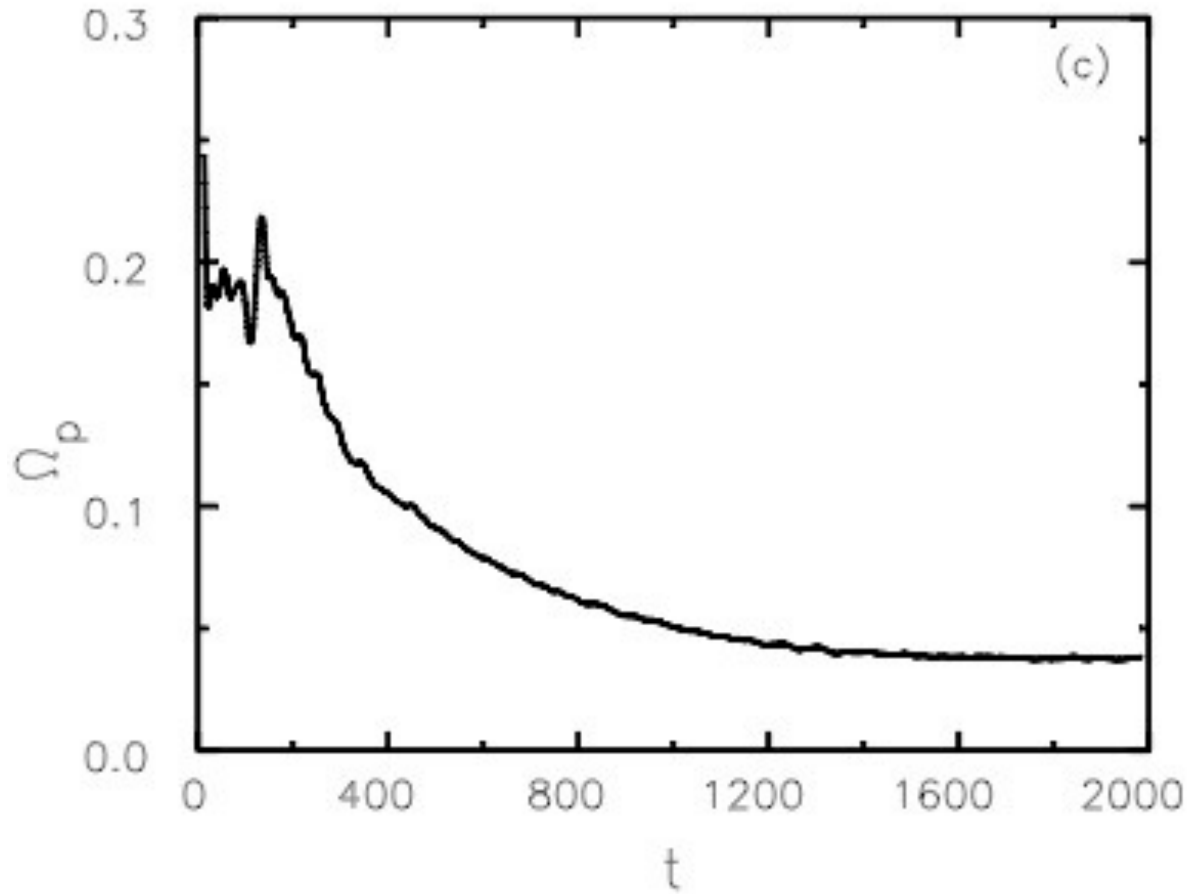
van den Bosch et al. 2000

Barred spirals



J. Gunn & Z. Frei

Barred spirals (cont.)



Debattista & Sellwood 2000

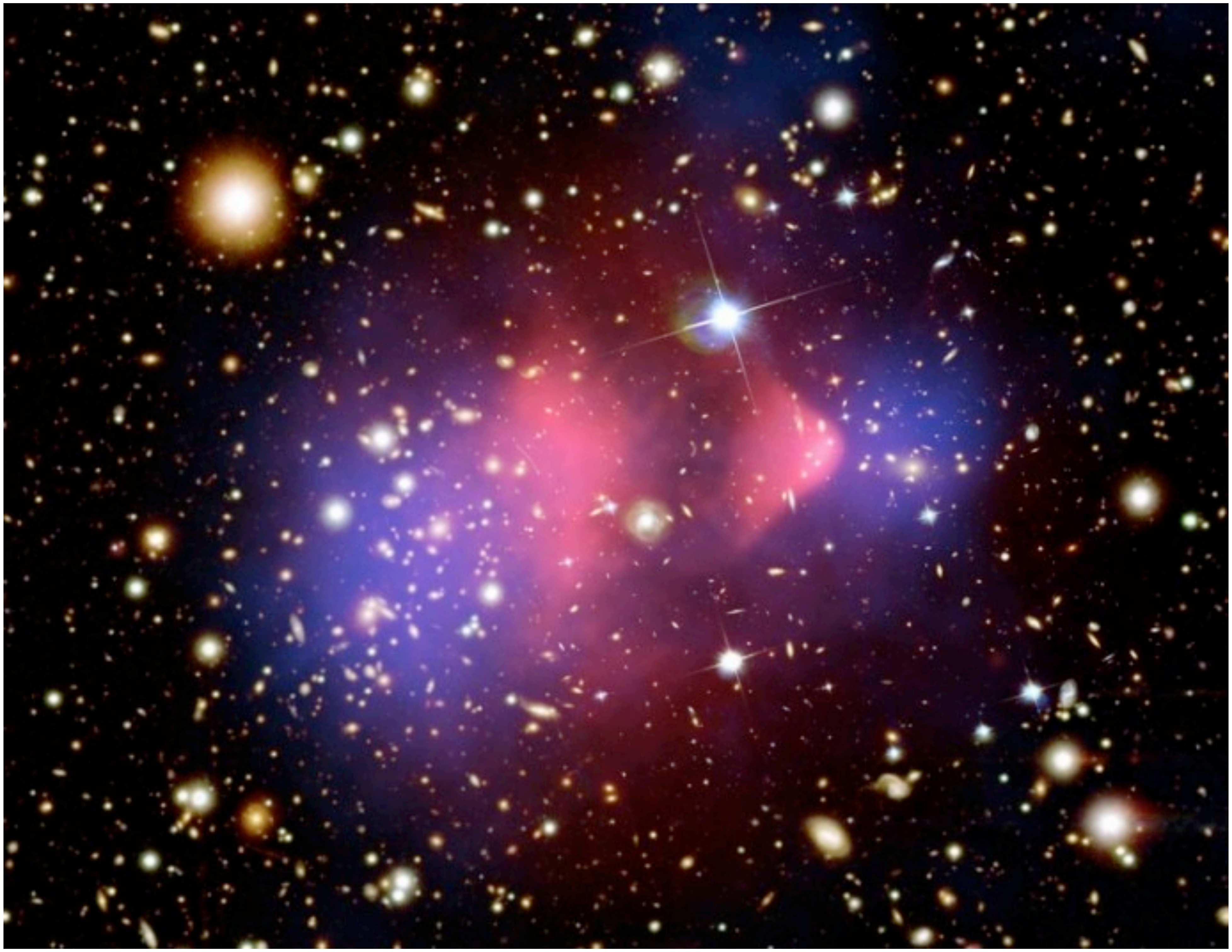
Repulsive Dark Matter*:
The Executive Summary

- Assumptions:
  - A light (1-100 eV) conserved boson
  - Born in a Bose-Einstein condensate (~axion)
  - Repulsive, short-range interactions
- Consequences:
  - Constant core radius for all central densities
  - Nontrivial thermodynamics, e.g. superfluidity
  - Reduced dynamical friction near the core
  - Acceptable high-z behavior
  - Hydrodynamic halos; ram-pressure stripping

*Peebles 2000; Goodman 2000; Goodman & Z. Slepian 2011

The more recent work on RDM, especially that concerning finite-temperature effects, has been done in collaboration with Zachary Slepian.

The bullet cluster



X-ray (*red*): NASA/CXC/CfA/ M.Markevitch et al.;
Lensing Map(*blue*): NASA/STScI; ESO WFI; Magellan/U.Arizona/ D.Clowe et al.

Also known as 1E-0657-56. *Red*: X-ray gas (viewed with *Chandra* Observatory, M. Markevitch et al.). *Blue:* Mass---including dark mass---reconstructed by gravitational lensing (D. Clowe et al.). This is perhaps the best evidence to date for the collisionless nature of dark matter, if this truly represents the result of a collision between clusters.

Energetics of exchange terms

1-particle momentum states in volume V: $\langle \boldsymbol{r}|\alpha\rangle = \frac{1}{\sqrt{V}}\exp\left(\frac{i\boldsymbol{p}_\alpha\cdot\boldsymbol{r}}{\hbar}\right)$; $\langle\alpha|\beta\rangle = \int\frac{d\boldsymbol{r}}{V}e^{i(\boldsymbol{p}_\alpha-\boldsymbol{p}_\beta)\cdot\boldsymbol{r}/\hbar} = \delta_{\alpha\beta}$

2 (or more) particles: $|1,2\rangle = \frac{|\alpha(1)\rangle|\beta(2)\rangle + |\alpha(2)\rangle|\beta(1)\rangle}{\sqrt{2(1+\delta_{\alpha\beta})}}$

2 − body potential: $U(\boldsymbol{r}_1-\boldsymbol{r}_2)$. Fourier transform: $\tilde{U}(\boldsymbol{q}) \equiv \int d\Delta\boldsymbol{r}\,U(\Delta\boldsymbol{r})\exp\left(\frac{-i\boldsymbol{q}\cdot\Delta\boldsymbol{r}}{\hbar}\right)$.

Interaction energy:

$$\langle 1,2|\mathrm{U}|1,2\rangle = \frac{1}{2\left(1+\delta_{\alpha\beta}\right)V}\left\{\tilde{U}(\boldsymbol{p}_\alpha-\boldsymbol{p}_\alpha)+\tilde{U}(\boldsymbol{p}_\alpha-\boldsymbol{p}_\beta)+\tilde{U}(\boldsymbol{p}_\beta-\boldsymbol{p}_\alpha)+\tilde{U}(\boldsymbol{p}_\beta-\boldsymbol{p}_\beta)\right\}$$

$$\approx \frac{\tilde{U}(0)}{\left(1+\delta_{\alpha\beta}\right)V} \quad \text{if the range of } U(\Delta\boldsymbol{r}) \text{ is } \textbf{short}: \ll \frac{\hbar}{\left|\boldsymbol{p}_\alpha-\boldsymbol{p}_\beta\right|}$$

A short-range interaction energy is <u>halved</u> between pairs of particles in the same momentum state.

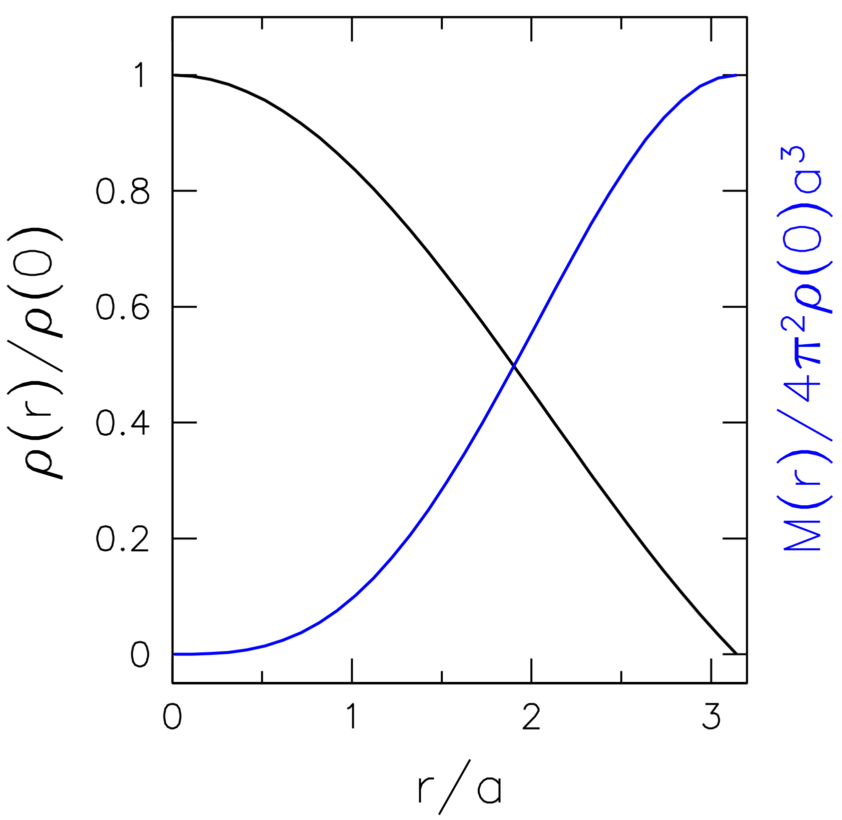
Pure condensate

Total energy of N particles in the same momentum state:

$$E = N\frac{\boldsymbol{p}_0^2}{2m} + \frac{N(N-1)}{2V}\tilde{U}(0); \qquad \frac{E}{N} \approx \frac{\boldsymbol{p}_0^2}{2m} + \chi, \quad \chi \equiv \frac{N}{2V}\tilde{U}(0)$$

Equation of State: $\boxed{P = K\rho^2}$,

$$P \equiv \text{pressure}, \;\; \rho \equiv m\frac{N}{V} = \text{mass density}, \quad K \equiv \frac{\tilde{U}(0)}{2m^2}$$



Emden polytrope of index $n=1$:

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM_r}{r^2} = -\frac{4\pi G}{r^2}\int_0^r d\bar{r}\,\rho(\bar{r})\bar{r}^2; \quad \frac{d\rho}{dr}(0) = 0.$$

$$\Rightarrow \rho(x) = \rho(0)\frac{\sin x}{x}, \quad x \equiv \frac{r}{a}$$

$$\boxed{a \equiv \sqrt{\frac{K}{2\pi G}} = \sqrt{\frac{\tilde{U}(0)}{4\pi G m^2}}}$$

Radius $r_c = \pi a$ independent of total mass.

Superfluidity

- Superfluidity follows from energetic penalties for exciting particles out of the condensate

Energy of a pure condensate: $E = N\left[\frac{N-1}{2V}\tilde{U}(0)+\frac{\boldsymbol{p}_0^2}{2m}\right]$. Let a single particle be excited

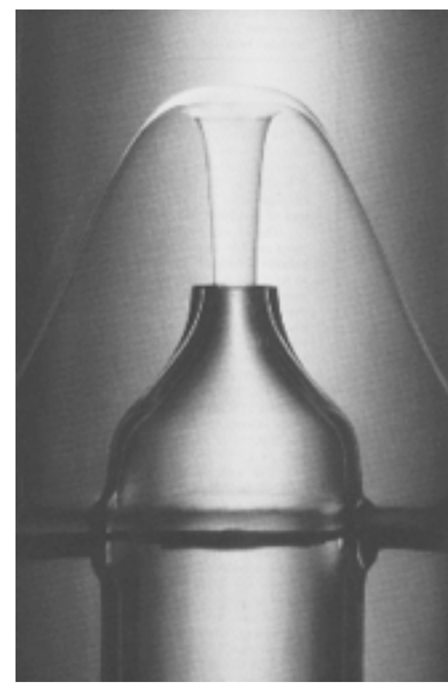
from $\boldsymbol{p}_0 \to \boldsymbol{p}'$: $E' = (N-1)\left[\frac{N-2}{2V}\tilde{U}(0)+\frac{\boldsymbol{p}_0^2}{2m}\right]+\frac{N-1}{V}\left[\tilde{U}(0)+\tilde{U}(\boldsymbol{p}'-\boldsymbol{p}_0)\right]+\frac{\boldsymbol{p}'^2-\boldsymbol{p}_0^2}{2m}$.

Then the penalty is: $\Delta E = \frac{N-1}{V}\tilde{U}(\boldsymbol{p}'-\boldsymbol{p}_0) + \frac{\boldsymbol{p}'^2-\boldsymbol{p}_0^2}{2m} \approx \frac{N}{V}\tilde{U}(0) + \frac{\boldsymbol{p}'^2-\boldsymbol{p}_0^2}{2m}$

- Critical velocity:

Consider scattering from a stationary potential ("spoon"):

$\Delta E = 0$ and $\boldsymbol{p}'^2 \geq 0 \;\Rightarrow\; \boxed{\frac{p_0}{m} \geq V_{\text{crit}} \equiv \sqrt{\frac{2N\tilde{U}(0)}{Vm}}}$



Dynamical friction in ideal fluids

Wave drag on a rigid obstacle of speed $\boldsymbol{V}$ **:**

1. $\omega_{\boldsymbol{k}} = \boldsymbol{k}\cdot\boldsymbol{V} \equiv kV\cos\theta$, i.e. waves are stationary in rest frame of perturber;
2. Dispersion relation.

E.g., sound waves: $\omega_{\boldsymbol{k}}^2 = c_s^2 k^2 \quad \Rightarrow \quad V\cos\theta = c_s \quad \Rightarrow \quad \boxed{V \geq c_s}$

In an infinite homogenous medium (with "Jeans swindle") $\omega_{\boldsymbol{k}}^2 = c_s^2 k^2 - 4\pi G\rho$

$\Rightarrow \quad k^2 = \dfrac{k_{\mathrm{J}}^2}{1 - M^2\cos^2\theta}, \quad M \equiv \dfrac{V}{c_s}, \quad k_{\mathrm{J}}^2 \equiv \dfrac{4\pi G\rho}{c_s^2}.$ So drag becomes possible for $M < 1$.

Drag involves spatial Fourier components of the perturbing potential (Φ):

$$\frac{dE}{dt} = \boldsymbol{F}_{\mathrm{drag}}\cdot\boldsymbol{V} \;=\; \rho\int\frac{d^3\boldsymbol{k}}{(2\pi)^2}\,\delta\big(\omega_{\boldsymbol{k}} - \boldsymbol{k}\cdot\boldsymbol{V}\big)\left|\boldsymbol{k}\tilde{\Phi}(\boldsymbol{k})\right|^2, \qquad \Phi\big(\boldsymbol{r} - \boldsymbol{V}t\big) = \int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\tilde{\Phi}(\boldsymbol{k})$$

NLSE & dispersion relation

NLSE: $i\hbar \frac{\partial \varphi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \varphi + \tilde{U}(0) |\varphi|^2 \varphi + \left(\Phi_{\text{ext}} + \frac{1}{2}\Phi_{\text{self}}\right)\varphi,$

$\mathcal{N} = |\varphi|^2$

$\nabla^2 \Phi_{\text{self}} = 4\pi G m \mathcal{N}$

Linearized perturbations : $\varphi = \varphi_0(t)\exp\left[\eta_R(\boldsymbol{r},t) + i\eta_I(\boldsymbol{r},t)\right], \quad \eta \ll 1,$

$|\varphi_0|^2 = \mathcal{N}_0 = constant. \qquad \eta(\boldsymbol{r},t) \rightarrow \eta(\boldsymbol{r})\exp(i\boldsymbol{k}\cdot\boldsymbol{r}).$

Dispersion relation:

$$\omega_{\boldsymbol{k}}^2 = c_s^2 k^2 - 4\pi G\rho + \underbrace{\frac{\hbar^2 k^4}{(2m)^2}}_{\text{negligible if } k \ll mc_s/\hbar}$$

The condensate responds to long-wavelength perturbations like an ideal compressible fluid.

RDM parameters

There are 3 (nonrelativistic) free parameters: $m, \tilde{U}(0)$, $\Omega_{\rm RDM} = \dfrac{8\pi G\bar{\rho}_{\rm RDM}}{3H_0^2}$

Constraints:

#1. $\Omega_{\rm RDM} \approx \Omega_{\rm m} \approx 0.27$

#2. $r_{\rm c} \equiv \pi a = \sqrt{\dfrac{\pi\tilde{U}(0)}{4Gm^2}} \sim 1$ kpc (?)

Scattering cross section: $\sigma = \dfrac{1}{\pi}\left[\dfrac{m\tilde{U}(0)}{\hbar^2}\right]^2 \approx 5\times 10^{-6}\left(\dfrac{mc^2}{100\,\text{eV}}\right)^6\left(\dfrac{r_{\rm c}}{\text{kpc}}\right)^4 \text{cm}^2$

Mean free path: $\lambda_{\rm mfp} = \dfrac{m}{\rho\sigma} \approx 2\times 10^{-2}\left(\dfrac{mc^2}{100\,\text{eV}}\right)^{-5}\left(\dfrac{\rho c^2}{\text{GeV cm}^{-3}}\right)^{-1}\left(\dfrac{r_{\rm c}}{\text{kpc}}\right)^{-4} \text{cm}$

Number density: $\nu = \dfrac{\rho}{m} \approx 10^7\left(\dfrac{mc^2}{100\,\text{eV}}\right)^{-1}\left(\dfrac{\rho c^2}{\text{GeV cm}^{-3}}\right)\text{cm}^{-3}$

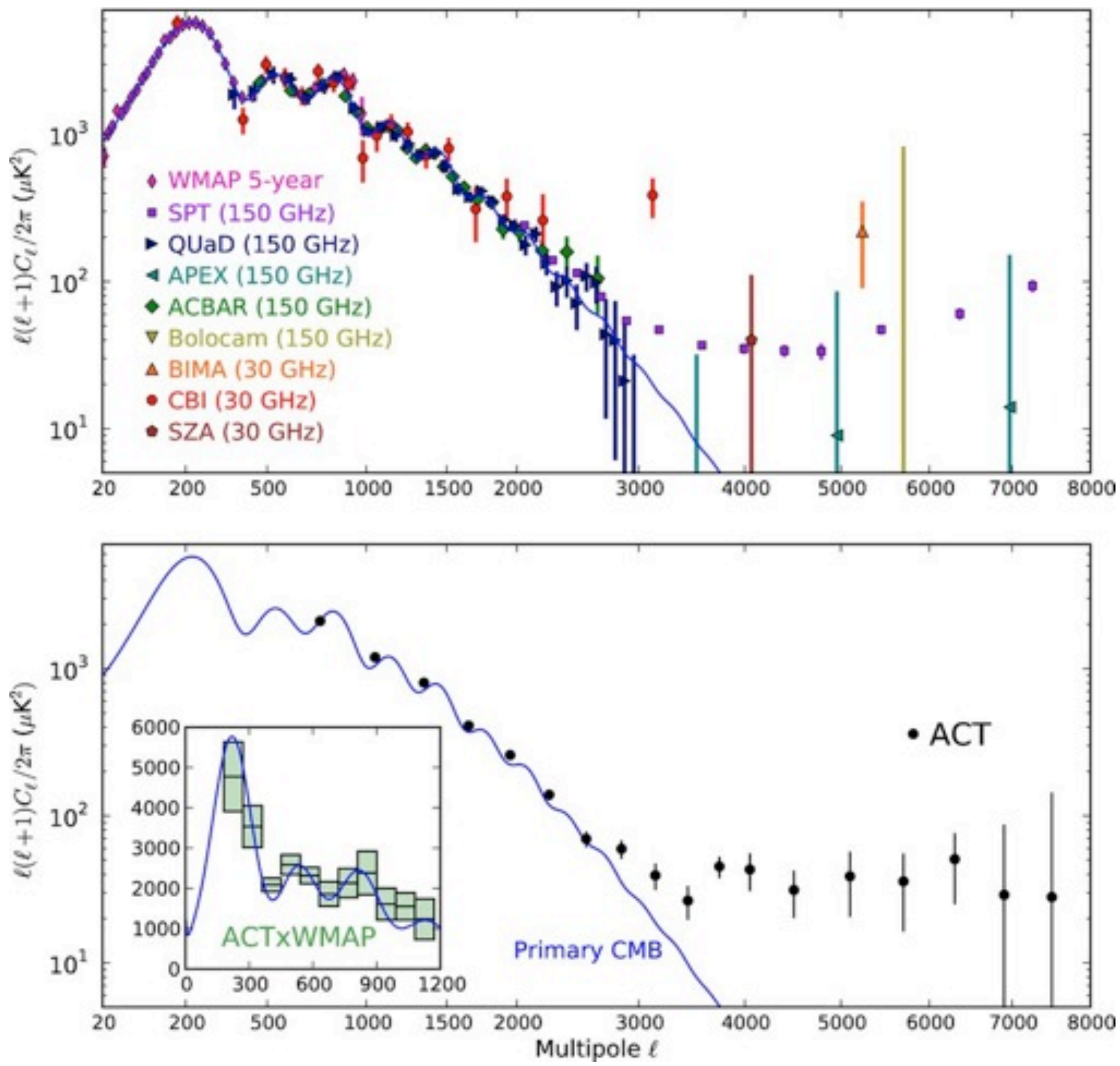
Other constraints

- Relativistic & early-universe effects
  - ✓ primordial neutrino synthesis & neutrino bounds
  - ✓ CMB fluctuations at high l
- Finite-temperature effects (w. Z. Slepian)
  - **Isothermal equilibria for galactic halos**
  - Interactions between merging halos: stripping

RDM and the CMB

$$\lambda_{\mathrm{Jeans}} = 2\pi a = 2r_{\mathrm{C}} \quad \forall z < z_{\mathrm{rel}} \sim 10^5$$

$$\ell_{\mathrm{J}} = \frac{2\pi d_A}{\lambda_{\mathrm{J}}} \approx 4\times 10^4 \, r_{\mathrm{c,kpc}}^{-1} \text{ at } z_{\mathrm{rec}} \approx 10^3$$



Fowler et al. 2010, ApJ 722, 1148

Relativistic EOS ($\rho \geq c^2/K$)

- A simple relativistic lagrangian for a scalar field ($\hbar = c = 1$):

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\phi\,\partial_\mu\phi + V(\phi), \qquad V(\phi) = \tfrac{1}{2}m^2\phi^2 + \kappa\phi^n$$

- The corresponding energy density is $\left[\varpi \equiv \partial\mathcal{L}/\partial(\partial_t\phi)\right]$

$$\mathcal{H} = \varpi\,\partial_t\phi - \mathcal{L} = \tfrac{1}{2}\varpi^2 + (\nabla\phi)^2 + V(\phi).$$

- Semiclassical action/volume ($\nabla\phi \approx 0$ in pure condensate):

$$\mathcal{I}(\mathcal{H}) = \frac{1}{2\pi}\oint \varpi\, d\phi = \frac{1}{2\pi}\oint \sqrt{2\left[\mathcal{H} - V(\phi)\right]}\, d\phi$$

- $\mathcal{H} \to$ energy/volume $\equiv \mathcal{E}$, $\mathcal{I} \to$ quanta/volume $\equiv \hbar\mathcal{N}$

Small $\phi \Rightarrow \quad \mathcal{E} \approx mc^2\mathcal{N} + \dfrac{3\hbar^3\kappa}{2m^2c}\mathcal{N}^2 \to \rho c^2 + K\rho^2$ with $K = \dfrac{3\hbar^3\kappa}{2m^4c}$

Large $\phi \Rightarrow \quad \mathcal{E} \approx 1.377\,\hbar c\,\kappa^{1/3}\mathcal{N}^{4/3}$

$$mc^2 = \left(\frac{3\pi\hbar^3c^7\kappa}{4Gr_c^2}\right)^{1/4} \approx 11\kappa^{1/4}\ \text{eV}$$

.... **provided n = 4**

Professor T. Padmanabhan has raised doubts whether there exists a sensible non-relativistic quantum-mechanical limit of a relativistic scalar field with a ϕ^4 interaction.

Little effect on primordial nucleosynthesis

Effect of RDM on expansion rate during the relativistic era* depends on r_c but not m:

$$\mathcal{E}_\gamma = \hbar c \frac{\pi^2}{15}\left(\frac{\pi^2}{2\zeta(3)}\right)^{4/3} \mathcal{N}_\gamma^{4/3}; \quad \mathcal{E}_{\rm RDM} = \hbar c \pi^2 \left(\frac{3}{\Gamma^2\left(1/4\right)}\right)^{4/3} \kappa^{1/3}\mathcal{N}_{\rm RDM}^{4/3}$$

$$\Rightarrow \frac{\mathcal{E}_{\rm RDM}}{\mathcal{E}_\gamma} = \frac{0.99983\ldots}{\pi}\kappa^{1/3}\left(\frac{\mathcal{N}_{\rm RDM}}{\mathcal{N}_\gamma}\right)^{4/3} \quad \propto \quad \kappa^{1/3}\left(\frac{m^{-1}\Omega_m}{T_{\rm CMB}^3}\right)^{4/3} \quad \text{if } \frac{\mathcal{N}_{\rm RDM}}{\mathcal{N}_\gamma} \text{ is conserved.}$$

But $\kappa / m^4 \propto K \propto r_c^4$.

Hence $$\boxed{\frac{\mathcal{E}_{\rm RDM}}{\mathcal{E}_\gamma} \approx 0.011\left(\frac{r_{\rm c}}{1\,{\rm kpc}}\right)^{2/3}\left(\frac{\Omega_m h^2}{0.135}\times\frac{2.725{\rm K}}{T_{\rm CMB}}\right)^{4/3}}$$

$$^* z_{\rm rel} \approx 0.96\times 10^5 \left(\frac{r_{\rm c}}{1\,{\rm kpc}}\right)^{-2/3}\left(\frac{\Omega_m h^2}{0.135}\right)^{-1/3}$$

Note that the redshift $z_{\rm rel}$ above which RDM becomes relativistic is independent of the particle mass *m* at fixed $r_{\rm c}$ because the pressure in the condensate is due to the repulsion rather than finite temperature. Finite temperatures and entropies are expected to develop at lower redshift as structures become nonlinear and RDM undergoes shocks.

RDM thermodynamics

$n_\alpha \equiv$ number of quanta with momentum $\boldsymbol{p}_\alpha$; $\quad \vec{n} \equiv (n_0, n_1, \ldots)$; $\quad \|\vec{n}\| \equiv \sum_\alpha n_\alpha$

$$E(\vec{n}, V) = \sum_\alpha \frac{\boldsymbol{p}_\alpha^2}{2m} n_\alpha + \frac{\tilde{U}(0)}{2V}\left(N^2 + \sum_\alpha \sum_{\beta \neq \alpha} n_\beta n_\alpha \right) = \sum_\alpha \frac{\boldsymbol{p}_\alpha^2}{2m} n_\alpha + \frac{\tilde{U}(0)}{2V}\left(2N^2 - \sum_\alpha n_\alpha^2 \right)$$

Canonical ensemble: $\quad Z(T, V, N) = \sum_{\|\vec{n}\| = N} \exp\left[-E(\vec{n}, V)/kT \right]$

Grand-canonical ": $\quad Z(T, V, \mu) = \sum_N Z(T, V, N) \exp\left(N\mu/kT \right)$

negative term impedes summing each n_α independently in GCE

The solution is to recognize that n_α^2/V can be neglected in the thermodynamic limit $N \propto V \to \infty$ [ITL] unless $n_\alpha \sim O(N)$, which can happen only in the ground state $\alpha = 0$, since

$$\frac{\boldsymbol{p}_\alpha^2}{2m} \gtrsim \frac{\hbar^2}{2m} V^{-2/3} \text{ when } \boldsymbol{p}_\alpha \neq \boldsymbol{p}_0 = \boldsymbol{0}$$

Details

$$\ln Z(T,V,N) = \ln Z_0(T,V,n_0) + \ln Z_{>0}(T,V,N'), \quad \text{where } N' \equiv N - n_0,$$

$$\ln Z_0(T,V,n_0) = \text{CE of ground state} = \frac{n_0(n_0+2N')}{2V^2}\tilde{U}(0),$$

$$\ln Z_{>0}(T,V,N') = \ln Z_{>0}(T,V,\mu') - \frac{\mu' N'}{kT} = \text{ CE of the excited (>0) states},$$

$$\mu' = \mu - \frac{N}{V}\tilde{U}(0) \equiv \text{governs particle exchange between ground \& excited states } \textit{at fixed } N,$$

$$\ln Z_{>0}(T,V,\mu') = V\left(\frac{2\pi mkT}{h^2}\right)^{3/2} \mathrm{Li}_{5/2}\left[\exp\left(\frac{\mu' - 2\chi}{kT}\right)\right], \qquad \chi \equiv \frac{N}{2V}\tilde{U}(0),$$

$$\mathrm{Li}_s(z) \equiv \text{ polylogarithm} = \frac{z}{\Gamma(s)}\int_0^\infty \frac{t^{s-1}dt}{e^t - z} = \sum_{n=0}^{\infty}\frac{z^n}{n^s}, \quad z<1$$

Results for EOS at finite T

Condensate occurs when $\frac{N}{V} \equiv v \geq v_{\text{crit}}(T) = \zeta\left(\tfrac{3}{2}\right)\left(\frac{2\pi mkT}{h^2}\right)^{3/2}$, as for ideal gas.

Dimensionless variables: $\hat{v} \equiv \frac{v}{v_{\text{crit}}}$, $\hat{v}_0 \equiv \frac{n_0/V}{v_{\text{crit}}}$, $\hat{P} \equiv \frac{P}{v_{\text{crit}}kT} \propto \frac{P}{T^{5/2}}$,

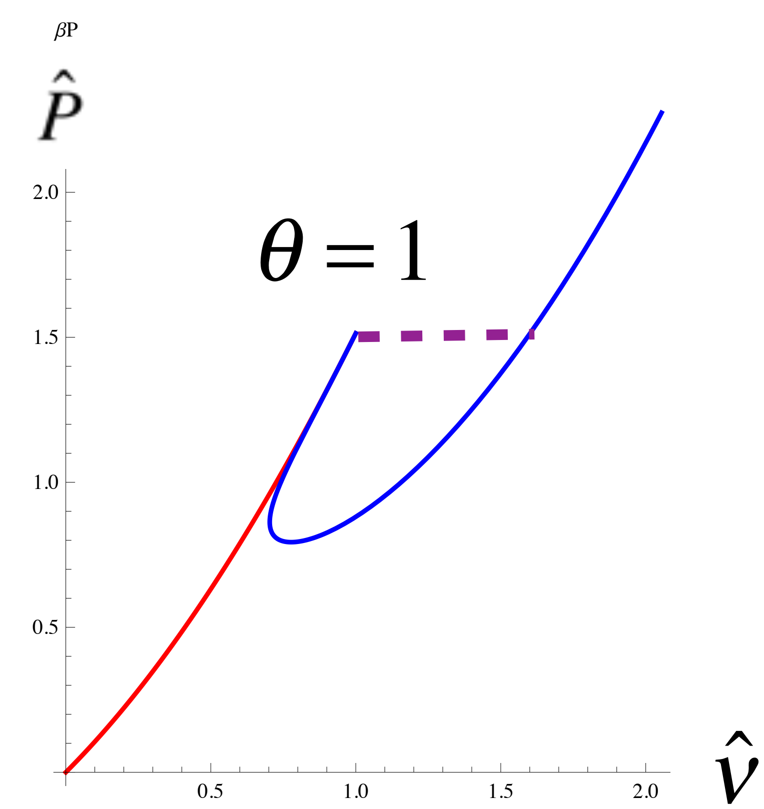
and $\theta \equiv \frac{\tilde{U}(0)}{kT} v_{\text{crit}} = \tfrac{1}{2}\zeta\left(\tfrac{3}{2}\right)\sqrt{\frac{\sigma}{\lambda_{\text{dB}}^2}}$, $\lambda_{\text{dB}} \equiv \frac{h}{\sqrt{2mkT}} =$ thermal de Broglie wavelength.

Require $\theta \ll 1$ for weak coupling $\Leftrightarrow$ well-defined single-particle momentum states.

$$\hat{P} \equiv \frac{\beta P}{v_{\text{crit}}} = \theta\left(\hat{v}^2 - \tfrac{1}{2}\hat{v}_0^2\right) + \zeta\left(\tfrac{3}{2}\right)^{-1}\text{Li}_{5/2}(z),$$

$$\hat{v} = \max(\hat{v}_0, 0) + \zeta\left(\tfrac{3}{2}\right)^{-1}\text{Li}_{3/2}(z),$$

$$z = \exp(-\theta\hat{v}_0) \quad \text{if} \quad \hat{v}_0 > 0.$$



The condensate is absent along the red branch of the pressure-vs.-density curve. At the cusp, condensate appears, and its density increases monotonically, but the total density and pressure are not monotonic below the purple dashed line. Hence, a first order phase transition occurs, but with a strength that is proportional to θ, which is << 1 in “realistic” models.

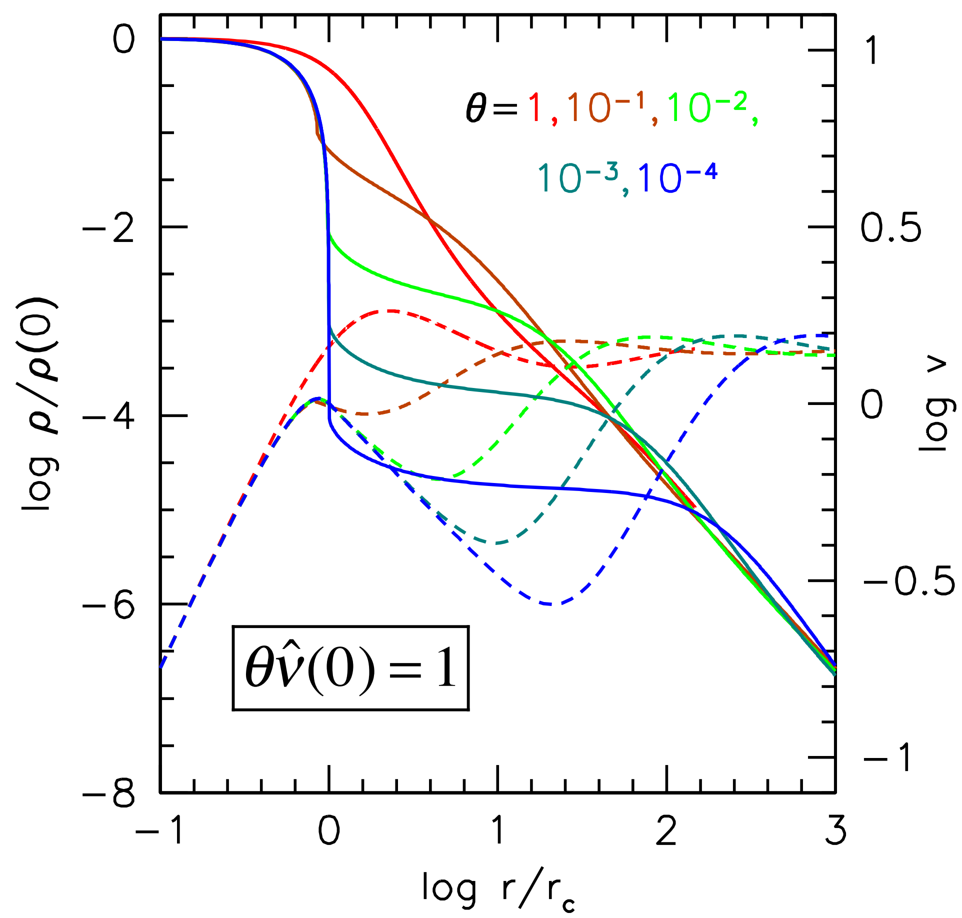
Isothermal RDM halos

- The shapes of the density profiles & rotation curves follow 2 parameters:

$$\hat{v}(0) = \frac{\rho(0)}{m\nu_{\text{crit}}}, \quad \theta \approx \frac{\sqrt{\sigma}}{\lambda_{\text{dB}}}$$

Note: $\hat{v}(0)\theta = \frac{8G\rho(0)}{v_\infty^2} r_c^2$

where $v_\infty^2 = \frac{2kT}{m} = \lim_{r\to\infty} \frac{GM(r)}{r}$



There is a shelf $\Delta \log \rho \sim -\log\theta$ at the core,

because $\rho_{\text{crit}} = \frac{\pi G v_\infty^2}{8 r_c^2}\theta.$

Thus we probably need $1 > \theta > 10^{-4}$ for a reasonable rotation curve

Summary

- Dynamical relaxation in collisionless gravitating systems, despite the firm foundation laid by Chandrasekhar, still has somewhat obscure relations to chaos and thermodynamics.
- Dynamical relaxation and dynamical friction involving dark matter may be modified its non-gravitational interactions, if any, and perhaps in surprising ways.